

The Coq Proof Assistant

The standard library

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πr^2 Project (formerly LogiCal, then TypiCal)

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Contents

This document is a short description of the COQ standard library. This library comes with the system as a complement of the core library (the **Init** library ; see the Reference Manual for a description of this library). It provides a set of modules directly available through the **Require** command.

The standard library is composed of the following subdirectories:

Logic Classical logic and dependent equality

Bool Booleans (basic functions and results)

Arith Basic Peano arithmetic

ZArith Basic integer arithmetic

Reals Classical Real Numbers and Analysis

Lists Monomorphic and polymorphic lists (basic functions and results), Streams (infinite sequences defined with co-inductive types)

Sets Sets (classical, constructive, finite, infinite, power set, etc.)

Relations Relations (definitions and basic results).

Sorting Sorted list (basic definitions and heapsort correctness).

Wellfounded Well-founded relations (basic results).

Program Tactics to deal with dependently-typed programs and their proofs.

Classes Standard type class instances on relations and Coq part of the setoid rewriting tactic.

Each of these subdirectories contains a set of modules, whose specifications (GALLINA files) have been roughly, and automatically, pasted in the following pages. There is also a version of this document in HTML format on the WWW, which you can access from the COQ home page at <http://coq.inria.fr/library>.

Chapter 1

Library **Coq.Init.Datatypes**

Set Implicit Arguments.

Require Import Notations.

Require Import Logic.

1.1 Datatypes with zero and one element

Empty_set is a datatype with no inhabitant

Inductive *Empty_set* : **Set** :=.

unit is a singleton datatype with sole inhabitant *tt*

Inductive *unit* : **Set** :=

tt : *unit*.

1.2 The boolean datatype

bool is the datatype of the boolean values *true* and *false*

Inductive *bool* : **Set** :=

| *true* : *bool*

| *false* : *bool*.

Add Printing *If bool*.

Delimit Scope *bool_scope* **with** *bool*.

Basic boolean operators

Definition *andb* (*b1 b2*:*bool*) : *bool* := *if b1 then b2 else false*.

Definition *orb* (*b1 b2*:*bool*) : *bool* := *if b1 then true else b2*.

Definition *implb* (*b1 b2*:*bool*) : *bool* := *if b1 then b2 else true*.

Definition *xorb* (*b1 b2*:*bool*) : *bool* :=

match b1, b2 with

| *true, true* => *false*

```

| true, false ⇒ true
| false, true ⇒ true
| false, false ⇒ false
end.
Definition negb (b:bool) := if b then false else true.
Infix "||" := orb : bool_scope.
Infix "&&" := andb : bool_scope.

Basic properties of andb
Lemma andb_prop : ∀ a b:bool, andb a b = true → a = true ∧ b = true.
Hint Resolve andb_prop: bool.
Lemma andb_true_intro :
  ∀ b1 b2:bool, b1 = true ∧ b2 = true → andb b1 b2 = true.
Hint Resolve andb_true_intro: bool.

Interpretation of booleans as propositions
Inductive eq_true : bool → Prop := is_eq_true : eq_true true.
Hint Constructors eq_true : eq_true.

Another way of interpreting booleans as propositions
Definition is_true b := b = true.

is_true can be activated as a coercion by (Local) Coercion is_true : bool >-> Sortclass.
Additional rewriting lemmas about eq_true
Lemma eq_true_ind_r :
  ∀ (P : bool → Prop) (b : bool), P b → eq_true b → P true.
Lemma eq_true_rec_r :
  ∀ (P : bool → Set) (b : bool), P b → eq_true b → P true.
Lemma eq_true_rect_r :
  ∀ (P : bool → Type) (b : bool), P b → eq_true b → P true.

The BoolSpec inductive will be used to relate a boolean value and two propositions corresponding
respectively to the true case and the false case. Interest: BoolSpec behave nicely with case and
destruct. See also Bool.reflect when Q = ¬P.
Inductive BoolSpec (P Q : Prop) : bool → Prop :=
| BoolSpecT : P → BoolSpec P Q true
| BoolSpecF : Q → BoolSpec P Q false.
Hint Constructors BoolSpec.

```

1.3 Peano natural numbers

nat is the datatype of natural numbers built from *O* and successor *S*; note that the constructor name is the letter O. Numbers in *nat* can be denoted using a decimal notation; e.g. `3%nat` abbreviates *S (S (S O))*

```
Inductive nat : Set :=
```

```

| O : nat
| S : nat → nat.

```

Delimit Scope *nat_scope* with *nat*.

1.4 Container datatypes

option A is the extension of *A* with an extra element *None*

```

Inductive option (A:Type) : Type :=
| Some : A → option A
| None : option A.

```

```

Definition option_map (A B:Type) (f:A→B) (o : option A) : option B :=
match o with
| Some a ⇒ @Some B (f a)
| None ⇒ @None B
end.

```

sum A B, written $A + B$, is the disjoint sum of *A* and *B*

```

Inductive sum (A B:Type) : Type :=
| inl : A → sum A B
| inr : B → sum A B.

```

Notation " $x + y$ " := (*sum x y*) : *type_scope*.

prod A B, written $A \times B$, is the product of *A* and *B*; the pair *pair A B a b* of *a* and *b* is abbreviated (*a, b*)

```

Inductive prod (A B:Type) : Type :=
pair : A → B → prod A B.

```

Add Printing Let *prod*.

Notation " $x * y$ " := (*prod x y*) : *type_scope*.

Notation "(x , y , .. , z)" := (*pair .. (pair x y) .. z*) : *core_scope*.

Section projections.

```

Context {A : Type} {B : Type}.

```

```

Definition fst (p:A × B) := match p with
| (x, y) ⇒ x
end.

```

```

Definition snd (p:A × B) := match p with
| (x, y) ⇒ y
end.

```

End projections.

Hint Resolve *pair inl inr*: *core*.

Lemma surjective_pairing :

```

∀ (A B:Type) (p:A × B), p = pair (fst p) (snd p).

```

Lemma injective_projections :

$\forall (A B:\text{Type}) (p1\ p2:A \times B),$
 $\text{fst } p1 = \text{fst } p2 \rightarrow \text{snd } p1 = \text{snd } p2 \rightarrow p1 = p2.$

Definition prod_uncurry $(A\ B\ C:\text{Type}) (f:\text{prod } A\ B \rightarrow C)$
 $(x:A) (y:B) : C := f (\text{pair } x\ y).$

Definition prod_curry $(A\ B\ C:\text{Type}) (f:A \rightarrow B \rightarrow C)$
 $(p:\text{prod } A\ B) : C := \text{match } p \text{ with}$
 $| \text{pair } x\ y \Rightarrow f\ x\ y$
end.

Polymorphic lists and some operations

Inductive list $(A : \text{Type}) : \text{Type} :=$
 $| \text{nil} : \text{list } A$
 $| \text{cons} : A \rightarrow \text{list } A \rightarrow \text{list } A.$

Infix "::" := cons (at level 60, right associativity) : list_scope.

Delimit Scope list_scope with list.

Local Open Scope list_scope.

Definition length $(A : \text{Type}) : \text{list } A \rightarrow \text{nat} :=$
 fix length l :=
 match l with
 $| \text{nil} \Rightarrow 0$
 $| _ :: l' \Rightarrow S (\text{length } l')$
end.

Concatenation of two lists

Definition app $(A : \text{Type}) : \text{list } A \rightarrow \text{list } A \rightarrow \text{list } A :=$
 fix app l m :=
 match l with
 $| \text{nil} \Rightarrow m$
 $| a :: l1 \Rightarrow a :: \text{app } l1\ m$
end.

Infix "++" := app (right associativity, at level 60) : list_scope.

1.5 The comparison datatype

Inductive comparison : Set :=
 $| \text{Eq} : \text{comparison}$
 $| \text{Lt} : \text{comparison}$
 $| \text{Gt} : \text{comparison}.$

Definition CompOpp $(r:\text{comparison}) :=$
 match r with
 $| \text{Eq} \Rightarrow \text{Eq}$
 $| \text{Lt} \Rightarrow \text{Gt}$

| Gt \Rightarrow Lt
end.

Lemma CompOpp_involutive : $\forall c, \text{CompOpp } (\text{CompOpp } c) = c$.

Lemma CompOpp_inj : $\forall c c', \text{CompOpp } c = \text{CompOpp } c' \rightarrow c = c'$.

Lemma CompOpp_iff : $\forall c c', \text{CompOpp } c = c' \leftrightarrow c = \text{CompOpp } c'$.

The *CompareSpec* inductive relates a *comparison* value with three propositions, one for each possible case. Typically, it can be used to specify a comparison function via some equality and order predicates. Interest: *CompareSpec* behave nicely with **case** and **destruct**.

Inductive CompareSpec (Peq Plt Pgt : Prop) : comparison \rightarrow Prop :=
| CompEq : Peq \rightarrow CompareSpec Peq Plt Pgt Eq
| CompLt : Plt \rightarrow CompareSpec Peq Plt Pgt Lt
| CompGt : Pgt \rightarrow CompareSpec Peq Plt Pgt Gt.

Hint Constructors CompareSpec.

For having clean interfaces after extraction, *CompareSpec* is declared in Prop. For some situations, it is nonetheless useful to have a version in Type. Interestingly, these two versions are equivalent.

Inductive CompareSpecT (Peq Plt Pgt : Prop) : comparison \rightarrow Type :=
| CompEqT : Peq \rightarrow CompareSpecT Peq Plt Pgt Eq
| CompLtT : Plt \rightarrow CompareSpecT Peq Plt Pgt Lt
| CompGtT : Pgt \rightarrow CompareSpecT Peq Plt Pgt Gt.

Hint Constructors CompareSpecT.

Lemma CompareSpec2Type : $\forall \text{Peq Plt Pgt } c,$
CompareSpec Peq Plt Pgt $c \rightarrow$ CompareSpecT Peq Plt Pgt c .

As an alternate formulation, one may also directly refer to predicates *eq* and *lt* for specifying a comparison, rather than fully-applied propositions. This *CompSpec* is now a particular case of *CompareSpec*.

Definition CompSpec {A} (eq lt : A \rightarrow A \rightarrow Prop)(x y:A) : comparison \rightarrow Prop :=
CompareSpec (eq x y) (lt x y) (lt y x).

Definition CompSpecT {A} (eq lt : A \rightarrow A \rightarrow Prop)(x y:A) : comparison \rightarrow Type :=
CompareSpecT (eq x y) (lt x y) (lt y x).

Hint Unfold CompSpec CompSpecT.

Lemma CompSpec2Type : $\forall A (eq lt:A \rightarrow A \rightarrow \text{Prop}) x y c,$
CompSpec eq lt x y $c \rightarrow$ CompSpecT eq lt x y c .

1.6 Misc Other Datatypes

identity A a is the family of datatypes on *A* whose sole non-empty member is the singleton datatype *identity A a a* whose sole inhabitant is denoted *refl_identity A a*

Inductive identity (A:Type) (a:A) : A \rightarrow Type :=
identity_refl : identity a a.

Hint Resolve identity_refl: core.

Identity type

Definition ID := $\forall A:\mathbf{Type}, A \rightarrow A$.

Definition id : ID := fun A x \Rightarrow x.

Definition IDProp := $\forall A:\mathbf{Prop}, A \rightarrow A$.

Definition idProp : IDProp := fun A x \Rightarrow x.

Chapter 2

Library Coq.Init.Logic_Type

This module defines type constructors for types in **Type** (*Datatypes.v* and *Logic.v* defined them for types in **Set**)

Set **Implicit Arguments**.

Require Import *Datatypes*.

Require Export *Logic*.

Negation of a type in **Type**

Definition *notT* (*A*:**Type**) := *A* → **False**.

Properties of *identity*

Section *identity_is_a_congruence*.

Variables *A B* : **Type**.

Variable *f* : *A* → *B*.

Variables *x y z* : *A*.

Lemma *identity_sym* : *identity* *x y* → *identity* *y x*.

Lemma *identity_trans* : *identity* *x y* → *identity* *y z* → *identity* *x z*.

Lemma *identity_congr* : *identity* *x y* → *identity* (*f* *x*) (*f* *y*).

Lemma *not_identity_sym* : *notT* (*identity* *x y*) → *notT* (*identity* *y x*).

End *identity_is_a_congruence*.

Definition *identity_ind_r* :

$\forall (A:\mathbf{Type}) (a:A) (P:A \rightarrow \mathbf{Prop}), P\ a \rightarrow \forall y:A, \mathit{identity}\ y\ a \rightarrow P\ y.$

Defined.

Definition *identity_rec_r* :

$\forall (A:\mathbf{Type}) (a:A) (P:A \rightarrow \mathbf{Set}), P\ a \rightarrow \forall y:A, \mathit{identity}\ y\ a \rightarrow P\ y.$

Defined.

Definition *identity_rect_r* :

$\forall (A:\mathbf{Type}) (a:A) (P:A \rightarrow \mathbf{Type}), P\ a \rightarrow \forall y:A, \mathit{identity}\ y\ a \rightarrow P\ y.$

Defined.

Hint Immediate *identity_sym not_identity_sym*: *core v62*.

```
Notation refl_id := identity_refl (compat "8.3").
Notation sym_id := identity_sym (compat "8.3").
Notation trans_id := identity_trans (compat "8.3").
Notation sym_not_id := not_identity_sym (compat "8.3").
```

Chapter 3

Library Coq.Init.Logic

Set Implicit Arguments.

Require Export Notations.

Notation "A -> B" := (\forall ($_$: A), B) : type_scope.

3.1 Propositional connectives

True is the always true proposition

Inductive True : Prop :=

| : True.

False is the always false proposition Inductive False : Prop :=.

not A, written $\neg A$, is the negation of A Definition not (A:Prop) := A \rightarrow False.

Notation "~ x" := (not x) : type_scope.

Hint Unfold not: core.

and A B, written $A \wedge B$, is the conjunction of A and B

conj p q is a proof of $A \wedge B$ as soon as p is a proof of A and q a proof of B

proj1 and *proj2* are first and second projections of a conjunction

Inductive and (A B:Prop) : Prop :=

conj : A \rightarrow B \rightarrow A \wedge B

where "A /\ B" := (and A B) : type_scope.

Section Conjunction.

Variables A B : Prop.

Theorem proj1 : A \wedge B \rightarrow A.

Theorem proj2 : A \wedge B \rightarrow B.

End Conjunction.

or A B, written $A \vee B$, is the disjunction of A and B

Inductive or (A B:Prop) : Prop :=

| or_introl : $A \rightarrow A \vee B$
| or_intror : $B \rightarrow A \vee B$

where "A \vee B" := (or A B) : type_scope.

iff A B, written $A \leftrightarrow B$, expresses the equivalence of A and B

Definition iff (A B:Prop) := (A \rightarrow B) \wedge (B \rightarrow A).

Notation "A \leftrightarrow B" := (iff A B) : type_scope.

Section Equivalence.

Theorem iff_refl : $\forall A:\text{Prop}, A \leftrightarrow A$.

Theorem iff_trans : $\forall A B C:\text{Prop}, (A \leftrightarrow B) \rightarrow (B \leftrightarrow C) \rightarrow (A \leftrightarrow C)$.

Theorem iff_sym : $\forall A B:\text{Prop}, (A \leftrightarrow B) \rightarrow (B \leftrightarrow A)$.

End Equivalence.

Hint Unfold iff: extcore.

Backward direction of the equivalences above does not need assumptions

Theorem and_iff_compat_l : $\forall A B C : \text{Prop},$
 $(B \leftrightarrow C) \rightarrow (A \wedge B \leftrightarrow A \wedge C)$.

Theorem and_iff_compat_r : $\forall A B C : \text{Prop},$
 $(B \leftrightarrow C) \rightarrow (B \wedge A \leftrightarrow C \wedge A)$.

Theorem or_iff_compat_l : $\forall A B C : \text{Prop},$
 $(B \leftrightarrow C) \rightarrow (A \vee B \leftrightarrow A \vee C)$.

Theorem or_iff_compat_r : $\forall A B C : \text{Prop},$
 $(B \leftrightarrow C) \rightarrow (B \vee A \leftrightarrow C \vee A)$.

Some equivalences

Theorem neg_false : $\forall A : \text{Prop}, \neg A \leftrightarrow (A \leftrightarrow \text{False})$.

Theorem and_cancel_l : $\forall A B C : \text{Prop},$
 $(B \rightarrow A) \rightarrow (C \rightarrow A) \rightarrow ((A \wedge B \leftrightarrow A \wedge C) \leftrightarrow (B \leftrightarrow C))$.

Theorem and_cancel_r : $\forall A B C : \text{Prop},$
 $(B \rightarrow A) \rightarrow (C \rightarrow A) \rightarrow ((B \wedge A \leftrightarrow C \wedge A) \leftrightarrow (B \leftrightarrow C))$.

Theorem and_comm : $\forall A B : \text{Prop}, A \wedge B \leftrightarrow B \wedge A$.

Theorem and_assoc : $\forall A B C : \text{Prop}, (A \wedge B) \wedge C \leftrightarrow A \wedge B \wedge C$.

Theorem or_cancel_l : $\forall A B C : \text{Prop},$
 $(B \rightarrow \neg A) \rightarrow (C \rightarrow \neg A) \rightarrow ((A \vee B \leftrightarrow A \vee C) \leftrightarrow (B \leftrightarrow C))$.

Theorem or_cancel_r : $\forall A B C : \text{Prop},$
 $(B \rightarrow \neg A) \rightarrow (C \rightarrow \neg A) \rightarrow ((B \vee A \leftrightarrow C \vee A) \leftrightarrow (B \leftrightarrow C))$.

Theorem or_comm : $\forall A B : \text{Prop}, (A \vee B) \leftrightarrow (B \vee A)$.

Theorem or_assoc : $\forall A B C : \text{Prop}, (A \vee B) \vee C \leftrightarrow A \vee B \vee C$.

Lemma iff_and : $\forall A B : \text{Prop}, (A \leftrightarrow B) \rightarrow (A \rightarrow B) \wedge (B \rightarrow A)$.

Lemma iff_to_and : $\forall A B : \text{Prop}, (A \leftrightarrow B) \leftrightarrow (A \rightarrow B) \wedge (B \rightarrow A)$.

$(IF_then_else\ P\ Q\ R)$, written $IF\ P\ then\ Q\ else\ R$ denotes either P and Q , or $\neg P$ and Q

Definition $IF_then_else\ (P\ Q\ R:Prop) := P \wedge Q \vee \neg P \wedge R$.

Notation "'IF' c1 'then' c2 'else' c3" := $(IF_then_else\ c1\ c2\ c3)$
(at level 200, right associativity) : *type_scope*.

3.2 First-order quantifiers

$ex\ P$, or simply $\exists x, P\ x$, or also $\exists x:A, P\ x$, expresses the existence of an x of some type A in **Set** which satisfies the predicate P . This is existential quantification.

$ex2\ P\ Q$, or simply $exists2\ x, P\ x \ \&\ Q\ x$, or also $exists2\ x:A, P\ x \ \&\ Q\ x$, expresses the existence of an x of type A which satisfies both predicates P and Q .

Universal quantification is primitively written $\forall x:A, Q$. By symmetry with existential quantification, the construction $all\ P$ is provided too.

Inductive $ex\ (A:Type)\ (P:A \rightarrow Prop) : Prop :=$
 $ex_intro : \forall x:A, P\ x \rightarrow ex\ (A:=A)\ P$.

Inductive $ex2\ (A:Type)\ (P\ Q:A \rightarrow Prop) : Prop :=$
 $ex_intro2 : \forall x:A, P\ x \rightarrow Q\ x \rightarrow ex2\ (A:=A)\ P\ Q$.

Definition $all\ (A:Type)\ (P:A \rightarrow Prop) := \forall x:A, P\ x$.

Notation "'exists' x .. y , p" := $(ex\ (fun\ x \Rightarrow .. (ex\ (fun\ y \Rightarrow p))\ ..))$
(at level 200, x binder, right associativity,
format "'[' 'exists' '/' 'x .. y , '/' 'p ']'")
: *type_scope*.

Notation "'exists2' x , p & q" := $(ex2\ (fun\ x \Rightarrow p)\ (fun\ x \Rightarrow q))$
(at level 200, x ident, p at level 200, right associativity) : *type_scope*.

Notation "'exists2' x : t , p & q" := $(ex2\ (fun\ x:t \Rightarrow p)\ (fun\ x:t \Rightarrow q))$
(at level 200, x ident, t at level 200, p at level 200, right associativity,
format "'[' 'exists2' '/' 'x : t , '/' '[' p & '/' q ']' ']'")
: *type_scope*.

Derived rules for universal quantification

Section *universal_quantification*.

Variable $A : Type$.

Variable $P : A \rightarrow Prop$.

Theorem $inst : \forall x:A, all\ (fun\ x \Rightarrow P\ x) \rightarrow P\ x$.

Theorem $gen : \forall (B:Prop) (f:\forall y:A, B \rightarrow P\ y), B \rightarrow all\ P$.

End *universal_quantification*.

3.3 Equality

$eq\ x\ y$, or simply $x=y$ expresses the equality of x and y . Both x and y must belong to the same type A . The definition is inductive and states the reflexivity of the equality. The others properties

(symmetry, transitivity, replacement of equals by equals) are proved below. The type of x and y can be made explicit using the notation $x = y :> A$. This is Leibniz equality as it expresses that x and y are equal iff every property on A which is true of x is also true of y

```
Inductive eq (A:Type) (x:A) : A → Prop :=
  eq_refl : x = x :> A
```

```
where "x = y :> A" := (@eq A x y) : type_scope.
```

```
Notation "x = y" := (x = y :>_) : type_scope.
```

```
Notation "x <> y :> T" := (¬ x = y :> T) : type_scope.
```

```
Notation "x <> y" := (x ≠ y :>_) : type_scope.
```

```
Hint Resolve I conj or_introl or_intror : core.
```

```
Hint Resolve eq_refl: core.
```

```
Hint Resolve ex_intro ex_intro2: core.
```

```
Section Logic_lemmas.
```

```
Theorem absurd : ∀ A C:Prop, A → ¬ A → C.
```

```
Section equality.
```

```
Variables A B : Type.
```

```
Variable f : A → B.
```

```
Variables x y z : A.
```

```
Theorem eq_sym : x = y → y = x.
```

```
Theorem eq_trans : x = y → y = z → x = z.
```

```
Theorem f_equal : x = y → f x = f y.
```

```
Theorem not_eq_sym : x ≠ y → y ≠ x.
```

```
End equality.
```

```
Definition eq_ind_r :
```

```
  ∀ (A:Type) (x:A) (P:A → Prop), P x → ∀ y:A, y = x → P y.
```

```
Defined.
```

```
Definition eq_rec_r :
```

```
  ∀ (A:Type) (x:A) (P:A → Set), P x → ∀ y:A, y = x → P y.
```

```
Defined.
```

```
Definition eq_rect_r :
```

```
  ∀ (A:Type) (x:A) (P:A → Type), P x → ∀ y:A, y = x → P y.
```

```
Defined.
```

```
End Logic_lemmas.
```

```
Module EQNOTATIONS.
```

```
Notation "'rew' H 'in' H'" := (eq_rect _ _ H' _ H)
```

```
  (at level 10, H' at level 10,
   format "'[' 'rew' H in '/' H' ']'").
```

```
Notation "'rew' [ P ] H 'in' H'" := (eq_rect _ P H' _ H)
```

```
  (at level 10, H' at level 10,
```



```

format "[ 'rew' [ P ] '/' H in '/' H' ]".
Notation "rew' <- H 'in' H'" := (eq_rect_r _ H' H)
(at level 10, H' at level 10,
format "[ 'rew' <- H in '/' H' ]").
Notation "rew' <- [ P ] H 'in' H'" := (eq_rect_r P H' H)
(at level 10, H' at level 10,
format "[ 'rew' <- [ P ] '/' H in '/' H' ]").
Notation "rew' -> H 'in' H'" := (eq_rect _ _ H' _ H)
(at level 10, H' at level 10, only parsing).
Notation "rew' -> [ P ] H 'in' H'" := (eq_rect _ P H' _ H)
(at level 10, H' at level 10, only parsing).

End EQNOTATIONS.

Import EqNotations.

Lemma rew_opp_r : ∀ A (P:A→Type) (x y:A) (H:x=y) (a:P y), rew H in rew ← H in a = a.
Lemma rew_opp_l : ∀ A (P:A→Type) (x y:A) (H:x=y) (a:P x), rew ← H in rew H in a = a.

Theorem f_equal2 :
  ∀ (A1 A2 B:Type) (f:A1 → A2 → B) (x1 y1:A1)
  (x2 y2:A2), x1 = y1 → x2 = y2 → f x1 x2 = f y1 y2.

Theorem f_equal3 :
  ∀ (A1 A2 A3 B:Type) (f:A1 → A2 → A3 → B) (x1 y1:A1)
  (x2 y2:A2) (x3 y3:A3),
  x1 = y1 → x2 = y2 → x3 = y3 → f x1 x2 x3 = f y1 y2 y3.

Theorem f_equal4 :
  ∀ (A1 A2 A3 A4 B:Type) (f:A1 → A2 → A3 → A4 → B)
  (x1 y1:A1) (x2 y2:A2) (x3 y3:A3) (x4 y4:A4),
  x1 = y1 → x2 = y2 → x3 = y3 → x4 = y4 → f x1 x2 x3 x4 = f y1 y2 y3 y4.

Theorem f_equal5 :
  ∀ (A1 A2 A3 A4 A5 B:Type) (f:A1 → A2 → A3 → A4 → A5 → B)
  (x1 y1:A1) (x2 y2:A2) (x3 y3:A3) (x4 y4:A4) (x5 y5:A5),
  x1 = y1 →
  x2 = y2 →
  x3 = y3 → x4 = y4 → x5 = y5 → f x1 x2 x3 x4 x5 = f y1 y2 y3 y4 y5.

Theorem f_equal_compose : ∀ A B C (a b:A) (f:A→B) (g:B→C) (e:a=b),
  f_equal g (f_equal f e) = f_equal (fun a => g (f a)) e.

The goupoid structure of equality

Theorem eq_trans_refl_l : ∀ A (x y:A) (e:x=y), eq_trans eq_refl e = e.
Theorem eq_trans_refl_r : ∀ A (x y:A) (e:x=y), eq_trans e eq_refl = e.
Theorem eq_sym_involutive : ∀ A (x y:A) (e:x=y), eq_sym (eq_sym e) = e.
Theorem eq_trans_sym_inv_l : ∀ A (x y:A) (e:x=y), eq_trans (eq_sym e) e = eq_refl.
Theorem eq_trans_sym_inv_r : ∀ A (x y:A) (e:x=y), eq_trans e (eq_sym e) = eq_refl.
Theorem eq_trans_assoc : ∀ A (x y z t:A) (e:x=y) (e':y=z) (e'':z=t),

```

$\text{eq_trans } e \text{ (eq_trans } e' \text{ } e'') = \text{eq_trans (eq_trans } e \text{ } e') \text{ } e''.$

Extra properties of equality

Theorem `eq_id_comm_l` : $\forall A (f:A \rightarrow A) (Hf:\forall a, a = f a), \forall a, \text{f_equal } f (Hf a) = Hf (f a).$

Theorem `eq_id_comm_r` : $\forall A (f:A \rightarrow A) (Hf:\forall a, f a = a), \forall a, \text{f_equal } f (Hf a) = Hf (f a).$

Lemma `eq_trans_map_distr` : $\forall A B x y z (f:A \rightarrow B) (e:x=y) (e':y=z), \text{f_equal } f (\text{eq_trans } e \text{ } e') = \text{eq_trans (f_equal } f \text{ } e) (\text{f_equal } f \text{ } e').$

Lemma `eq_sym_map_distr` : $\forall A B (x y:A) (f:A \rightarrow B) (e:x=y), \text{eq_sym (f_equal } f \text{ } e) = \text{f_equal } f (\text{eq_sym } e).$

Lemma `eq_trans_sym_distr` : $\forall A (x y z:A) (e:x=y) (e':y=z), \text{eq_sym (eq_trans } e \text{ } e') = \text{eq_trans (eq_sym } e') (\text{eq_sym } e).$

Notation `sym_eq` := `eq_sym (compat "8.3")`.

Notation `trans_eq` := `eq_trans (compat "8.3")`.

Notation `sym_not_eq` := `not_eq_sym (compat "8.3")`.

Notation `refl_equal` := `eq_refl (compat "8.3")`.

Notation `sym_equal` := `eq_sym (compat "8.3")`.

Notation `trans_equal` := `eq_trans (compat "8.3")`.

Notation `sym_not_equal` := `not_eq_sym (compat "8.3")`.

Hint Immediate `eq_sym not_eq_sym`: *core*.

Basic definitions about relations and properties

Definition `subrelation` ($A B : \text{Type}$) ($R R' : A \rightarrow B \rightarrow \text{Prop}$) :=
 $\forall x y, R x y \rightarrow R' x y.$

Definition `unique` ($A : \text{Type}$) ($P : A \rightarrow \text{Prop}$) ($x:A$) :=
 $P x \wedge \forall (x':A), P x' \rightarrow x=x'.$

Definition `uniqueness` ($A:\text{Type}$) ($P:A \rightarrow \text{Prop}$) := $\forall x y, P x \rightarrow P y \rightarrow x = y.$

Unique existence

Notation "'exists' ! x .. y , p" :=
 $(\text{ex (unique (fun } x \Rightarrow \dots (\text{ex (unique (fun } y \Rightarrow p)))) \dots))$
 $(\text{at level 200, } x \text{ binder, right associativity,}$
 $\text{format "['exists' ! ' / ' x .. y , ' / ' p ']'")$
 $: \text{type_scope}.$

Lemma `unique_existence` : $\forall (A:\text{Type}) (P:A \rightarrow \text{Prop}),$
 $((\exists x, P x) \wedge \text{uniqueness } P) \leftrightarrow (\exists! x, P x).$

Lemma `forall_exists_unique_domain_coincide` :
 $\forall A (P:A \rightarrow \text{Prop}), (\exists! x, P x) \rightarrow$
 $\forall Q:A \rightarrow \text{Prop}, (\forall x, P x \rightarrow Q x) \leftrightarrow (\exists x, P x \wedge Q x).$

Lemma `forall_exists_coincide_unique_domain` :
 $\forall A (P:A \rightarrow \text{Prop}),$
 $(\forall Q:A \rightarrow \text{Prop}, (\forall x, P x \rightarrow Q x) \leftrightarrow (\exists x, P x \wedge Q x))$
 $\rightarrow (\exists! x, P x).$

3.4 Being inhabited

The predicate *inhabited* can be used in different contexts. If A is thought as a type, *inhabited* A states that A is inhabited. If A is thought as a computationally relevant proposition, then *inhabited* A weakens A so as to hide its computational meaning. The so-weakened proof remains computationally relevant but only in a propositional context.

Inductive *inhabited* (A :Type) : Prop := inhabits : $A \rightarrow$ *inhabited* A .

Hint Resolve *inhabits*: core.

Lemma exists_inhabited : $\forall (A:\text{Type}) (P:A \rightarrow \text{Prop}),$
 $(\exists x, P\ x) \rightarrow$ *inhabited* A .

Declaration of *stepl* and *stepr* for *eq* and *iff*

Lemma eq_stepl : $\forall (A : \text{Type}) (x\ y\ z : A), x = y \rightarrow x = z \rightarrow z = y$.

Declare Left Step eq_stepl.

Declare Right Step eq_trans.

Lemma iff_stepl : $\forall A\ B\ C : \text{Prop}, (A \leftrightarrow B) \rightarrow (A \leftrightarrow C) \rightarrow (C \leftrightarrow B)$.

Declare Left Step iff_stepl.

Declare Right Step iff_trans.

Chapter 4

Library **Coq.Init.Notations**

These are the notations whose level and associativity are imposed by Coq

Notations for propositional connectives

Reserved Notation " $x \rightarrow y$ " (at level 99, right associativity, y at level 200).

Reserved Notation " $x \leftrightarrow y$ " (at level 95, no associativity).

Reserved Notation " $x \wedge y$ " (at level 80, right associativity).

Reserved Notation " $x \vee y$ " (at level 85, right associativity).

Reserved Notation " $\sim x$ " (at level 75, right associativity).

Notations for equality and inequalities

Reserved Notation " $x = y :> T$ "

(at level 70, y at *next level*, no associativity).

Reserved Notation " $x = y$ " (at level 70, no associativity).

Reserved Notation " $x = y = z$ "

(at level 70, no associativity, y at *next level*).

Reserved Notation " $x <> y :> T$ "

(at level 70, y at *next level*, no associativity).

Reserved Notation " $x <> y$ " (at level 70, no associativity).

Reserved Notation " $x \leq y$ " (at level 70, no associativity).

Reserved Notation " $x < y$ " (at level 70, no associativity).

Reserved Notation " $x \geq y$ " (at level 70, no associativity).

Reserved Notation " $x > y$ " (at level 70, no associativity).

Reserved Notation " $x \leq y \leq z$ " (at level 70, y at *next level*).

Reserved Notation " $x \leq y < z$ " (at level 70, y at *next level*).

Reserved Notation " $x < y < z$ " (at level 70, y at *next level*).

Reserved Notation " $x < y \leq z$ " (at level 70, y at *next level*).

Arithmetical notations (also used for type constructors)

Reserved Notation " $x + y$ " (at level 50, left associativity).

Reserved Notation " $x - y$ " (at level 50, left associativity).

Reserved Notation " $x * y$ " (at level 40, left associativity).

Reserved Notation " x / y " (at level 40, left associativity).

Reserved Notation " $- x$ " (at level 35, right associativity).

Reserved Notation $/ x$ (at level 35, right associativity).

Reserved Notation $x \wedge y$ (at level 30, right associativity).

Notations for booleans

Reserved Notation $x \parallel y$ (at level 50, left associativity).

Reserved Notation $x \&\& y$ (at level 40, left associativity).

Notations for pairs

Reserved Notation (x, y, \dots, z) (at level 0).

Notation $\{x\}$ is reserved and has a special status as component of other notations such as $\{A\} + \{B\}$ and $A + \{B\}$ (which are at the same level as $x + y$); $\{x\}$ is at level 0 to factor with $\{x : A \mid P\}$

Reserved Notation $\{x\}$ (at level 0, x at level 99).

Notations for sigma-types or subsets

Reserved Notation $\{x \mid P\}$ (at level 0, x at level 99).

Reserved Notation $\{x \mid P \& Q\}$ (at level 0, x at level 99).

Reserved Notation $\{x : A \mid P\}$ (at level 0, x at level 99).

Reserved Notation $\{x : A \mid P \& Q\}$ (at level 0, x at level 99).

Reserved Notation $\{x : A \& P\}$ (at level 0, x at level 99).

Reserved Notation $\{x : A \& P \& Q\}$ (at level 0, x at level 99).

Delimit Scope *type_scope* with *type*.

Delimit Scope *core_scope* with *core*.

Open Scope *core_scope*.

Open Scope *type_scope*.

ML Tactic Notations

Chapter 5

Library `Coq.Init.Peano`

The type *nat* of Peano natural numbers (built from *O* and *S*) is defined in *Datatypes.v*. This module defines the following operations on natural numbers :

- predecessor *pred*
- addition *plus*
- multiplication *mult*
- less or equal order *le*
- less *lt*
- greater or equal *ge*
- greater *gt*

It states various lemmas and theorems about natural numbers, including Peano's axioms of arithmetic (in Coq, these are provable). Case analysis on *nat* and induction on $\text{nat} \times \text{nat}$ are provided too

```
Require Import Notations.
Require Import Datatypes.
Require Import Logic.
Require Coq.Init.Nat.

Open Scope nat_scope.

Definition eq_S := f_equal S.
Definition f_equal_nat := f_equal (A:=nat).

Hint Resolve eq_S: v62.
Hint Resolve f_equal_nat: core.
```

The predecessor function

```
Notation pred := Nat.pred (compat "8.4").
Definition f_equal_pred := f_equal pred.
```

Hint Resolve *f_equal_pred*: *v62*.

Theorem *pred_Sn* : $\forall n:\text{nat}, n = \text{pred } (\text{S } n)$.

Injectivity of successor

Definition *eq_add_S* $n\ m\ (H: \text{S } n = \text{S } m): n = m := \text{f_equal pred } H$.

Hint Immediate *eq_add_S*: *core*.

Theorem *not_eq_S* : $\forall n\ m:\text{nat}, n \neq m \rightarrow \text{S } n \neq \text{S } m$.

Hint Resolve *not_eq_S*: *core*.

Definition *IsSucc* $(n:\text{nat}) : \text{Prop} :=$

```
match n with
| 0 => False
| S p => True
end.
```

Zero is not the successor of a number

Theorem *O_S* : $\forall n:\text{nat}, 0 \neq \text{S } n$.

Hint Resolve *O_S*: *core*.

Theorem *n_Sn* : $\forall n:\text{nat}, n \neq \text{S } n$.

Hint Resolve *n_Sn*: *core*.

Addition

Notation *plus* := *Nat.add* (*compat* "8.4").

Infix "+" := *Nat.add* : *nat_scope*.

Definition *f_equal2_plus* := *f_equal2 plus*.

Hint Resolve *f_equal2_plus*: *v62*.

Definition *f_equal2_nat* := *f_equal2* (*A1*:=*nat*) (*A2*:=*nat*).

Hint Resolve *f_equal2_nat*: *core*.

Lemma *plus_n_O* : $\forall n:\text{nat}, n = n + 0$.

Hint Resolve *plus_n_O*: *core*.

Lemma *plus_O_n* : $\forall n:\text{nat}, 0 + n = n$.

Lemma *plus_n_Sm* : $\forall n\ m:\text{nat}, \text{S } (n + m) = n + \text{S } m$.

Hint Resolve *plus_n_Sm*: *core*.

Lemma *plus_Sn_m* : $\forall n\ m:\text{nat}, \text{S } n + m = \text{S } (n + m)$.

Standard associated names

Notation *plus_0_r_reverse* := *plus_n_O* (*compat* "8.2").

Notation *plus_succ_r_reverse* := *plus_n_Sm* (*compat* "8.2").

Multiplication

Notation *mult* := *Nat.mul* (*compat* "8.4").

Infix "×" := *Nat.mul* : *nat_scope*.

Definition *f_equal2_mult* := *f_equal2 mult*.

Hint Resolve *f_equal2_mult*: *core*.

Lemma *mult_n_O* : $\forall n:\text{nat}, 0 = n \times 0$.

Hint Resolve *mult_n_O*: *core*.

Lemma *mult_n_Sm* : $\forall n m:\text{nat}, n \times m + n = n \times S m$.

Hint Resolve *mult_n_Sm*: *core*.

Standard associated names

Notation *mult_0_r_reverse* := *mult_n_O* (*compat* "8.2").

Notation *mult_succ_r_reverse* := *mult_n_Sm* (*compat* "8.2").

Truncated subtraction: $m - n$ is 0 if $n \geq m$

Notation *minus* := *Nat.sub* (*compat* "8.4").

Infix "-" := *Nat.sub* : *nat_scope*.

Definition of the usual orders, the basic properties of *le* and *lt* can be found in files *Le* and *Lt*

Inductive *le* ($n:\text{nat}$) : $\text{nat} \rightarrow \text{Prop}$:=

| *le_n* : $n \leq n$

| *le_S* : $\forall m:\text{nat}, n \leq m \rightarrow n \leq S m$

where "n <= m" := (*le* n m) : *nat_scope*.

Hint Constructors *le*: *core*.

Definition *lt* ($n m:\text{nat}$) := $S n \leq m$.

Hint Unfold *lt*: *core*.

Infix "<" := *lt* : *nat_scope*.

Definition *ge* ($n m:\text{nat}$) := $m \leq n$.

Hint Unfold *ge*: *core*.

Infix "≥" := *ge* : *nat_scope*.

Definition *gt* ($n m:\text{nat}$) := $m < n$.

Hint Unfold *gt*: *core*.

Infix ">" := *gt* : *nat_scope*.

Notation "x <= y <= z" := $(x \leq y \wedge y \leq z)$: *nat_scope*.

Notation "x <= y < z" := $(x \leq y \wedge y < z)$: *nat_scope*.

Notation "x < y < z" := $(x < y \wedge y < z)$: *nat_scope*.

Notation "x < y <= z" := $(x < y \wedge y \leq z)$: *nat_scope*.

Theorem *le_pred* : $\forall n m, n \leq m \rightarrow \text{pred } n \leq \text{pred } m$.

Theorem *le_S_n* : $\forall n m, S n \leq S m \rightarrow n \leq m$.

Theorem *le_0_n* : $\forall n, 0 \leq n$.

Theorem *le_n_S* : $\forall n m, n \leq m \rightarrow S n \leq S m$.

Case analysis

Theorem *nat_case* :

$\forall (n:\text{nat}) (P:\text{nat} \rightarrow \text{Prop}), P 0 \rightarrow (\forall m:\text{nat}, P (S m)) \rightarrow P n$.

Principle of double induction

Theorem *nat_double_ind* :

$\forall R:\text{nat} \rightarrow \text{nat} \rightarrow \text{Prop},$
 $(\forall n:\text{nat}, R\ 0\ n) \rightarrow$
 $(\forall n:\text{nat}, R\ (\text{S } n)\ 0) \rightarrow$
 $(\forall n\ m:\text{nat}, R\ n\ m \rightarrow R\ (\text{S } n)\ (\text{S } m)) \rightarrow \forall n\ m:\text{nat}, R\ n\ m.$

Maximum and minimum : definitions and specifications

Notation `max` := `Nat.max` (*compat* "8.4").

Notation `min` := `Nat.min` (*compat* "8.4").

Lemma `max_l` $n\ m : m \leq n \rightarrow \text{Nat.max } n\ m = n.$

Lemma `max_r` $n\ m : n \leq m \rightarrow \text{Nat.max } n\ m = m.$

Lemma `min_l` $n\ m : n \leq m \rightarrow \text{Nat.min } n\ m = n.$

Lemma `min_r` $n\ m : m \leq n \rightarrow \text{Nat.min } n\ m = m.$

Lemma `nat_rect_succ_r` $\{A\} (f: A \rightarrow A) (x:A) n :$

`nat_rect` (`fun _ \Rightarrow A`) x (`fun _ \Rightarrow f`) (`S n`) = `nat_rect` (`fun _ \Rightarrow A`) ($f\ x$) (`fun _ \Rightarrow f`) n .

Theorem `nat_rect_plus` :

$\forall (n\ m:\text{nat}) \{A\} (f:A \rightarrow A) (x:A),$

`nat_rect` (`fun _ \Rightarrow A`) x (`fun _ \Rightarrow f`) ($n + m$) =

`nat_rect` (`fun _ \Rightarrow A`) (`nat_rect` (`fun _ \Rightarrow A`) x (`fun _ \Rightarrow f`) m) (`fun _ \Rightarrow f`) n .

Chapter 6

Library **Coq.Init.Prelude**

```
Require Export Notations.  
Require Export Logic.  
Require Export Logic_Type.  
Require Export Datatypes.  
Require Export Specif.  
Require Coq.Init.Nat.  
Require Export Peano.  
Require Export Coq.Init.Wf.  
Require Export Coq.Init.Tactics.  
Add Search Blacklist "_subproof" "Private_".
```

Chapter 7

Library **Coq.Init.Specif**

Basic specifications : sets that may contain logical information

Set Implicit Arguments.

Require Import Notations.

Require Import Datatypes.

Require Import Logic.

Subsets and Sigma-types

$(\text{sig } A \ P)$, or more suggestively $\{x:A \mid P \ x\}$, denotes the subset of elements of the type A which satisfy the predicate P . Similarly $(\text{sig2 } A \ P \ Q)$, or $\{x:A \mid P \ x \ \& \ Q \ x\}$, denotes the subset of elements of the type A which satisfy both P and Q .

Inductive $\text{sig } (A:\text{Type}) \ (P:A \rightarrow \text{Prop}) : \text{Type} :=$
 $\text{exist} : \forall x:A, P \ x \rightarrow \text{sig } P.$

Inductive $\text{sig2 } (A:\text{Type}) \ (P \ Q:A \rightarrow \text{Prop}) : \text{Type} :=$
 $\text{exist2} : \forall x:A, P \ x \rightarrow Q \ x \rightarrow \text{sig2 } P \ Q.$

$(\text{sigT } A \ P)$, or more suggestively $\{x:A \ \& \ (P \ x)\}$ is a Sigma-type. Similarly for $(\text{sigT2 } A \ P \ Q)$, also written $\{x:A \ \& \ (P \ x) \ \& \ (Q \ x)\}$.

Inductive $\text{sigT } (A:\text{Type}) \ (P:A \rightarrow \text{Type}) : \text{Type} :=$
 $\text{existT} : \forall x:A, P \ x \rightarrow \text{sigT } P.$

Inductive $\text{sigT2 } (A:\text{Type}) \ (P \ Q:A \rightarrow \text{Type}) : \text{Type} :=$
 $\text{existT2} : \forall x:A, P \ x \rightarrow Q \ x \rightarrow \text{sigT2 } P \ Q.$

Notation " $\{ x \mid P \}$ " := $(\text{sig } (\text{fun } x \Rightarrow P)) : \text{type_scope}.$

Notation " $\{ x \mid P \ \& \ Q \}$ " := $(\text{sig2 } (\text{fun } x \Rightarrow P) \ (\text{fun } x \Rightarrow Q)) : \text{type_scope}.$

Notation " $\{ x : A \mid P \}$ " := $(\text{sig } (A:=A) \ (\text{fun } x \Rightarrow P)) : \text{type_scope}.$

Notation " $\{ x : A \mid P \ \& \ Q \}$ " := $(\text{sig2 } (A:=A) \ (\text{fun } x \Rightarrow P) \ (\text{fun } x \Rightarrow Q)) :$
 $\text{type_scope}.$

Notation " $\{ x : A \ \& \ P \}$ " := $(\text{sigT } (A:=A) \ (\text{fun } x \Rightarrow P)) : \text{type_scope}.$

Notation " $\{ x : A \ \& \ P \ \& \ Q \}$ " := $(\text{sigT2 } (A:=A) \ (\text{fun } x \Rightarrow P) \ (\text{fun } x \Rightarrow Q)) :$
 $\text{type_scope}.$

Add Printing Let $\text{sig}.$

Add Printing Let $\text{sig}2$.
Add Printing Let $\text{sig}T$.
Add Printing Let $\text{sig}T2$.

Projections of *sig*

An element y of a subset $\{x:A \mid (P\ x)\}$ is the pair of an a of type A and of a proof h that a satisfies P . Then $(proj1_sig\ y)$ is the witness a and $(proj2_sig\ y)$ is the proof of $(P\ a)$

Section Subset_projections.

Variable A : Type.

Variable $P : A \rightarrow \text{Prop}$.

```

Definition proj1_sig (e:sig P) := match e with
| exist _ a b => a
end.

```

```

Definition proj2_sig (e:sig P) :=
  match e return P (proj1_sig e) with
  | exist _ a b => b
  end.

```

End Subset_projections.

sig2 of a predicate can be projected to a *sig*.

This allows *proj1-sig* and *proj2-sig* to be usable with *sig2*.

The **let** statements occur in the body of the *exist* so that *proj1_sig* of a coerced $X : sig2 \ P \ Q$ will unify with **let** $(a, -, -) := X$ in a

```

Definition sig_of_sig2 (A : Type) (P Q : A → Prop) (X : sig2 P Q) : sig P
:= exist P
    (let (a, -, -) := X in a)
    (let (x, p, -) as s return (P (let (a, -, -) := s in a)) := X in p).

```

Projections of *sig2*

An element y of a subset $\{x:A \mid (P\ x) \ \& \ (Q\ x)\}$ is the triple of an a of type A , a of a proof h that a satisfies P , and a proof h' that a satisfies Q . Then $(proj1\text{-}sig\ (sig\text{-}of\text{-}sig2\ y))$ is the witness a , $(proj2\text{-}sig\ (sig\text{-}of\text{-}sig2\ y))$ is the proof of $(P\ a)$, and $(proj3\text{-}sig\ y)$ is the proof of $(Q\ a)$.

Section Subset_projections2.

Variable A : Type.

Variables $P \ Q : A \rightarrow \text{Prop}$.

```

Definition proj3_sig (e : sig2 P Q) :=
  let (a, b, c) return Q (proj1_sig (sig_of_sig2 e)) := e in c.

```

End Subset_projections2.

Projections of *sigT*

An element x of a sigma-type $\{y:A \ \& \ P \ y\}$ is a dependent pair made of an a of type A and an h of type $P \ a$. Then, $(projT1 \ x)$ is the first projection and $(projT2 \ x)$ is the second projection, the type of which depends on the $projT1$.

Section Projections.

Variable A : Type.

Variable $P : A \rightarrow \text{Type}$.

Definition $\text{projT1} (x:\text{sigT } P) : A := \text{match } x \text{ with}$
 $\quad | \text{existT } _ \ a \ _ \Rightarrow a$
 end.

Definition $\text{projT2} (x:\text{sigT } P) : P (\text{projT1 } x) :=$
 $\quad \text{match } x \text{ return } P (\text{projT1 } x) \text{ with}$
 $\quad | \text{existT } _ \ _ \ h \Rightarrow h$
 end.

End Projections.

sigT2 of a predicate can be projected to a sigT .

This allows projT1 and projT2 to be usable with sigT2 .

The **let** statements occur in the body of the existT so that projT1 of a coerced $X : \text{sigT2 } P \ Q$ will unify with **let** $(a, -, -) := X$ in a

Definition $\text{sigT_of_sigT2} (A : \text{Type}) (P \ Q : A \rightarrow \text{Type}) (X : \text{sigT2 } P \ Q) : \text{sigT } P$
 $:= \text{existT } P$
 $\quad (\text{let } (a, -, -) := X \text{ in } a)$
 $\quad (\text{let } (x, p, -) \text{ as } s \text{ return } (P (\text{let } (a, -, -) := s \text{ in } a)) := X \text{ in } p).$

Projections of sigT2

An element x of a sigma-type $\{y:A \ \& \ P \ y \ \& \ Q \ y\}$ is a dependent pair made of an a of type A , an h of type $P \ a$, and an h' of type $Q \ a$. Then, $(\text{projT1 } (\text{sigT_of_sigT2 } x))$ is the first projection, $(\text{projT2 } (\text{sigT_of_sigT2 } x))$ is the second projection, and $(\text{projT3 } x)$ is the third projection, the types of which depends on the projT1 .

Section Projections2.

Variable $A : \text{Type}$.

Variables $P \ Q : A \rightarrow \text{Type}$.

Definition $\text{projT3} (e : \text{sigT2 } P \ Q) :=$
 $\quad \text{let } (a, b, c) \text{ return } Q (\text{projT1 } (\text{sigT_of_sigT2 } e)) := e \text{ in } c.$

End Projections2.

sigT of a predicate is equivalent to sig

Definition $\text{sig_of_sigT} (A : \text{Type}) (P : A \rightarrow \text{Prop}) (X : \text{sigT } P) : \text{sig } P$
 $:= \text{exist } P (\text{projT1 } X) (\text{projT2 } X).$

Definition $\text{sigT_of_sig} (A : \text{Type}) (P : A \rightarrow \text{Prop}) (X : \text{sig } P) : \text{sigT } P$
 $:= \text{existT } P (\text{proj1_sig } X) (\text{proj2_sig } X).$

sigT2 of a predicate is equivalent to sig2

Definition $\text{sig2_of_sigT2} (A : \text{Type}) (P \ Q : A \rightarrow \text{Prop}) (X : \text{sigT2 } P \ Q) : \text{sig2 } P \ Q$
 $:= \text{exist2 } P \ Q (\text{projT1 } (\text{sigT_of_sigT2 } X)) (\text{projT2 } (\text{sigT_of_sigT2 } X)) (\text{projT3 } X).$

Definition $\text{sigT2_of_sig2} (A : \text{Type}) (P \ Q : A \rightarrow \text{Prop}) (X : \text{sig2 } P \ Q) : \text{sigT2 } P \ Q$
 $:= \text{existT2 } P \ Q (\text{proj1_sig } (\text{sig_of_sig2 } X)) (\text{proj2_sig } (\text{sig_of_sig2 } X)) (\text{proj3_sig } X).$

sumbool is a boolean type equipped with the justification of their value

Inductive $\text{sumbool} (A \ B:\text{Prop}) : \text{Set} :=$

```

| left : A → {A} + {B}
| right : B → {A} + {B}
where "{ A } + { B }" := (sumbool A B) : type_scope.
Add Printing If sumbool.

```

sumor is an option type equipped with the justification of why it may not be a regular value

```

Inductive sumor (A:Type) (B:Prop) : Type :=
| inleft : A → A + {B}
| inright : B → A + {B}
where "A + { B }" := (sumor A B) : type_scope.
Add Printing If sumor.

```

Various forms of the axiom of choice for specifications

Section Choice_lemmas.

```

Variables S S' : Set.
Variable R : S → S' → Prop.
Variable R' : S → S' → Set.
Variables R1 R2 : S → Prop.

Lemma Choice :
  (∀ x:S, {y:S' | R x y}) → {f:S → S' | ∀ z:S, R z (f z)}.

Lemma Choice2 :
  (∀ x:S, {y:S' & R' x y}) → {f:S → S' & ∀ z:S, R' z (f z)}.

Lemma bool_choice :
  (∀ x:S, {R1 x} + {R2 x}) →
  {f:S → bool | ∀ x:S, f x = true ∧ R1 x ∨ f x = false ∧ R2 x}.

```

End Choice_lemmas.

Section Dependent_choice_lemmas.

```

Variables X : Set.
Variable R : X → X → Prop.

Lemma dependent_choice :
  (∀ x:X, {y | R x y}) →
  ∀ x0, {f : nat → X | f 0 = x0 ∧ ∀ n, R (f n) (f (S n))}.

```

End Dependent_choice_lemmas.

A result of type (*Exc A*) is either a normal value of type *A* or an *error* :

Inductive *Exc* [*A:Type*] : Type := *value* : *A* → (*Exc A*) | *error* : (*Exc A*).

It is implemented using the option type. Section *Exc*.

```

Variable A : Type.

Definition Exc := option A.
Definition value := @Some A.
Definition error := @None A.

```

End *Exc*.

```

Definition except := False_rec.

Theorem absurd_set :  $\forall (A:\text{Prop}) (C:\text{Set}), A \rightarrow \neg A \rightarrow C$ .

Hint Resolve left right inleft inright: core v62.
Hint Resolve exist exist2 existT existT2: core.

Notation sigS := sigT (compat "8.2").
Notation existS := existT (compat "8.2").
Notation sigS_rect := sigT_rect (compat "8.2").
Notation sigS_rec := sigT_rec (compat "8.2").
Notation sigS_ind := sigT_ind (compat "8.2").
Notation projS1 := projT1 (compat "8.2").
Notation projS2 := projT2 (compat "8.2").

Notation sigS2 := sigT2 (compat "8.2").
Notation existS2 := existT2 (compat "8.2").
Notation sigS2_rect := sigT2_rect (compat "8.2").
Notation sigS2_rec := sigT2_rec (compat "8.2").
Notation sigS2_ind := sigT2_ind (compat "8.2").

```

Chapter 8

Library `Coq.Init.Tactics`

```
Require Import Notations.  
Require Import Logic.  
Require Import Specif.
```

8.1 Useful tactics

Ex falso quodlibet : a tactic for proving `False` instead of the current goal. This is just a nicer name for tactics such as `elimtype False` and other `cut False`.

```
Ltac exfalso := elimtype False.
```

A tactic for proof by contradiction. With contradict `H`,

- $H: \neg A \vdash B$ gives $\vdash A$
- $H: \neg A \vdash \neg B$ gives $H: B \vdash A$
- $H: A \vdash B$ gives $\vdash \neg A$
- $H: A \vdash \neg B$ gives $H: B \vdash \neg A$
- $H: \text{False}$ leads to a resolved subgoal.

Moreover, negations may be in unfolded forms, and `A` or `B` may live in `Type`

```
Ltac contradict H :=  
  let save tac H := let x := fresh in intro x; tac H; rename x into H  
  in  
  let negpos H := case H; clear H  
  in  
  let negneg H := save negpos H  
  in  
  let pospos H :=  
    let A := type of H in (exfalso; revert H; try fold ( $\neg A$ ))  
  in  
  let posneg H := save pospos H
```



```

in
let neg H := match goal with
|  $\vdash (\neg \_)$   $\Rightarrow$  negneg H
|  $\vdash (\_ \rightarrow \text{False})$   $\Rightarrow$  negneg H
|  $\vdash \_ \Rightarrow$  negpos H
end in
let pos H := match goal with
|  $\vdash (\neg \_)$   $\Rightarrow$  posneg H
|  $\vdash (\_ \rightarrow \text{False})$   $\Rightarrow$  posneg H
|  $\vdash \_ \Rightarrow$  pospos H
end in
match type of H with
|  $(\neg \_)$   $\Rightarrow$  neg H
|  $(\_ \rightarrow \text{False})$   $\Rightarrow$  neg H
|  $\_ \Rightarrow$  (elim H;fail) || pos H
end.

Ltac swap H :=
  idtac "swap is OBSOLETE: use contradict instead.";
  intro; apply H; clear H.

Ltac absurd_hyp H :=
  idtac "absurd_hyp is OBSOLETE: use contradict instead.";
  let T := type of H in
  absurd T.

Ltac false_hyp H G :=
  let T := type of H in absurd T; [ apply G | assumption ].

Ltac case_eq x := generalize (eq_refl x); pattern x at -1; case x.

Ltac destr_eq H := discriminate H || (try (injection H; clear H; intro H)).

Tactic Notation "destruct_with_eqn" constr(x) :=
  destruct x eqn:?.
Tactic Notation "destruct_with_eqn" ident(n) :=
  try intros until n; destruct n eqn:?.
Tactic Notation "destruct_with_eqn" ":" ident(H) constr(x) :=
  destruct x eqn:H.
Tactic Notation "destruct_with_eqn" ":" ident(H) ident(n) :=
  try intros until n; destruct n eqn:H.

  Break every hypothesis of a certain type

Ltac destruct_all t :=
  match goal with
  |  $x : t \vdash \_ \Rightarrow$  destruct x; destruct_all t
  |  $\_ \Rightarrow$  idtac
  end.

```

Tactic Notation "rewrite_all" **constr**(*eq*) := repeat rewrite *eq* in *.

Tactic Notation "rewrite_all" "<-" **constr**(*eq*) := repeat rewrite \leftarrow *eq* in *.

Tactics for applying equivalences.

The following code provides tactics “apply \rightarrow t”, “apply \leftarrow t”, “apply \rightarrow t in H” and “apply \leftarrow t in H”. Here t is a term whose type consists of nested dependent and nondependent products with an equivalence $A \leftrightarrow B$ as the conclusion. The tactics with “ \rightarrow ” in their names apply $A \rightarrow B$ while those with “ \leftarrow ” in the name apply $B \rightarrow A$.

```
Ltac find_equiv H :=
let T := type of H in
lazymatch T with
| ?A  $\rightarrow$  ?B  $\Rightarrow$ 
  let H1 := fresh in
  let H2 := fresh in
  cut A;
  [intro H1; pose proof (H H1) as H2; clear H H1;
   rename H2 into H; find_equiv H |
   clear H]
|  $\forall$  x : ?t, _  $\Rightarrow$ 
  let a := fresh "a" with
    H1 := fresh "H" in
    evar (a : t); pose proof (H a) as H1; unfold a in H1;
    clear a; clear H; rename H1 into H; find_equiv H
| ?A  $\leftrightarrow$  ?B  $\Rightarrow$  idtac
| _  $\Rightarrow$  fail "The given statement does not seem to end with an equivalence."
end.
```

```
Ltac bapply lemma todo :=
let H := fresh in
  pose proof lemma as H;
  find_equiv H; [todo H; clear H | .. ].
```

Tactic Notation "apply" " \rightarrow " **constr**(*lemma*) :=
bapply lemma ltac:(fun H \Rightarrow destruct H as [H _]; apply H).

Tactic Notation "apply" " \leftarrow " **constr**(*lemma*) :=
bapply lemma ltac:(fun H \Rightarrow destruct H as [_ H]; apply H).

Tactic Notation "apply" " \rightarrow " **constr**(*lemma*) "in" *hyp*(*J*) :=
bapply lemma ltac:(fun H \Rightarrow destruct H as [H _]; apply H in J).

Tactic Notation "apply" " \leftarrow " **constr**(*lemma*) "in" *hyp*(*J*) :=
bapply lemma ltac:(fun H \Rightarrow destruct H as [_ H]; apply H in J).

An experimental tactic simpler than auto that is useful for ending proofs “in one step”

```
Ltac easy :=
let rec use_hyp H :=
  match type of H with
  | _  $\wedge$  _  $\Rightarrow$  exact H || destruct_hyp H
  | _  $\Rightarrow$  try solve [inversion H]
```

```

    end
with do_intro := let H := fresh in intro H; use_hyp H
with destruct_hyp H := case H; clear H; do_intro; do_intro in
let rec use_hyps :=
  match goal with
  | H : _ ∧ _ ⊢ _ ⇒ exact H || (destruct_hyp H; use_hyps)
  | H : _ ⊢ _ ⇒ solve [inversion H]
  | _ ⇒ idtac
  end in
let do_atom :=
  solve [ trivial with eq_true | reflexivity | symmetry; trivial | contradiction ] in
let rec do_ccl :=
  try do_atom;
  repeat (do_intro; try do_atom);
  solve [ split; do_ccl ] in
solve [ do_atom | use_hyps; do_ccl ] ||
fail "Cannot solve this goal".

Tactic Notation "now" tactic(t) := t; easy.

  Slightly more than easy

Ltac easy' := repeat split; simpl; easy || now destruct 1.

  A tactic to document or check what is proved at some point of a script

Ltac now_show c := change c.

  Support for rewriting decidability statements

Set Implicit Arguments.

Lemma decide_left : ∀ (C:Prop) (decide:{C}+{¬C}),
  C → ∀ P:{C}+{¬C}→Prop, (∀ H:C, P (left _ H)) → P decide.

Lemma decide_right : ∀ (C:Prop) (decide:{C}+{¬C}),
  ¬C → ∀ P:{C}+{¬C}→Prop, (∀ H:¬C, P (right _ H)) → P decide.

Tactic Notation "decide" constr(lemma) "with" constr(H) :=
  let try_to_merge_hyps H :=
    try (clear H; intro H) ||
    (let H' := fresh H "bis" in intro H'; try clear H') ||
    (let H' := fresh in intro H'; try clear H') in
  match type of H with
  | ¬ ?C ⇒ apply (decide_right lemma H); try_to_merge_hyps H
  | ?C → False ⇒ apply (decide_right lemma H); try_to_merge_hyps H
  | _ ⇒ apply (decide_left lemma H); try_to_merge_hyps H
  end.

  Clear an hypothesis and its dependencies

Tactic Notation "clear" "dependent" hyp(h) :=
  let rec depclear h :=
    clear h ||

```

```

match goal with
|  $H$  : context [  $h$  ]  $\vdash$  _  $\Rightarrow$  depclear  $H$ ; depclear  $h$ 
end ||
fail "hypothesis to clear is used in the conclusion (maybe indirectly)"
in depclear  $h$ .

```

Revert an hypothesis and its dependencies : this is actually generalize dependent...

```

Tactic Notation "revert" "dependent" hyp( $h$ ) :=
generalize dependent  $h$ .

```

Chapter 9

Library `Coq.Init.Wf`

9.1 This module proves the validity of

- well-founded recursion (also known as course of values)
- well-founded induction

from a well-founded ordering on a given set

```
Set Implicit Arguments.
```

```
Require Import Notations.
```

```
Require Import Logic.
```

```
Require Import Datatypes.
```

Well-founded induction principle on `Prop`

```
Section Well_founded.
```

```
Variable A : Type.
```

```
Variable R : A → A → Prop.
```

The accessibility predicate is defined to be non-informative (`Acc_rect` is automatically defined because `Acc` is a singleton type)

```
Inductive Acc (x: A) : Prop :=
```

```
  Acc_intro : (∀ y:A, R y x → Acc y) → Acc x.
```

```
Lemma Acc_inv : ∀ x:A, Acc x → ∀ y:A, R y x → Acc y.
```

```
Global Implicit Arguments Acc_inv [x y] [x].
```

A relation is well-founded if every element is accessible

```
Definition well_founded := ∀ a:A, Acc a.
```

Well-founded induction on `Set` and `Prop`

```
Hypothesis Rwf : well_founded.
```

```
Theorem well_founded_induction_type :
```

```
  ∀ P:A → Type,  
  (∀ x:A, (∀ y:A, R y x → P y) → P x) → ∀ a:A, P a.
```

Theorem *well_founded_induction* :

$\forall P:A \rightarrow \mathbf{Set},$
 $(\forall x:A, (\forall y:A, R\ y\ x \rightarrow P\ y) \rightarrow P\ x) \rightarrow \forall a:A, P\ a.$

Theorem *well_founded_ind* :

$\forall P:A \rightarrow \mathbf{Prop},$
 $(\forall x:A, (\forall y:A, R\ y\ x \rightarrow P\ y) \rightarrow P\ x) \rightarrow \forall a:A, P\ a.$

Well-founded fixpoints

Section *FixPoint*.

Variable $P : A \rightarrow \mathbf{Type}.$

Variable $F : \forall x:A, (\forall y:A, R\ y\ x \rightarrow P\ y) \rightarrow P\ x.$

Fixpoint *Fix_F* $(x:A) (a:\mathbf{Acc}\ x) : P\ x :=$

$F\ (\mathbf{fun}\ (y:A) (h:R\ y\ x) \Rightarrow \mathbf{Fix_F}\ (\mathbf{Acc_inv}\ a\ h)).$

Scheme *Acc_inv_dep* := **Induction for** *Acc* **Sort** *Prop*.

Lemma *Fix_F_eq* :

$\forall (x:A) (r:\mathbf{Acc}\ x),$
 $F\ (\mathbf{fun}\ (y:A) (p:R\ y\ x) \Rightarrow \mathbf{Fix_F}\ (x:=y) (\mathbf{Acc_inv}\ r\ p)) = \mathbf{Fix_F}\ (x:=x)\ r.$

Definition *Fix* $(x:A) := \mathbf{Fix_F}\ (Rwf\ x).$

Proof that *well_founded_induction* satisfies the fixpoint equation. It requires an extra property of the functional

Hypothesis

F_ext :
 $\forall (x:A) (f\ g:\forall y:A, R\ y\ x \rightarrow P\ y),$
 $(\forall (y:A) (p:R\ y\ x), f\ y\ p = g\ y\ p) \rightarrow F\ f = F\ g.$

Lemma *Fix_F_inv* : $\forall (x:A) (r\ s:\mathbf{Acc}\ x), \mathbf{Fix_F}\ r = \mathbf{Fix_F}\ s.$

Lemma *Fix_eq* : $\forall x:A, \mathbf{Fix}\ x = F\ (\mathbf{fun}\ (y:A) (p:R\ y\ x) \Rightarrow \mathbf{Fix}\ y).$

End *FixPoint*.

End *Well_founded*.

Well-founded fixpoints over pairs

Section *Well_founded_2*.

Variables $A\ B : \mathbf{Type}.$

Variable $R : A \times B \rightarrow A \times B \rightarrow \mathbf{Prop}.$

Variable $P : A \rightarrow B \rightarrow \mathbf{Type}.$

Section *FixPoint_2*.

Variable

$F :$
 $\forall (x:A) (x':B),$
 $(\forall (y:A) (y':B), R\ (y, y')\ (x, x') \rightarrow P\ y\ y') \rightarrow P\ x\ x'.$

Fixpoint *Fix_F_2* $(x:A) (x':B) (a:\mathbf{Acc}\ R\ (x, x')) : P\ x\ x' :=$

F

(**fun** (y:A) (y':B) (h:R (y, y') (x, x')) \Rightarrow
 Fix_F_2 (x:=y) (x':=y') (Acc_inv a (y,y') h)).

End FixPoint_2.

Hypothesis Rwf : well_founded R.

Theorem well_founded_induction_type_2 :

(\forall (x:A) (x':B),
 (\forall (y:A) (y':B), R (y, y') (x, x') \rightarrow P y y') \rightarrow P x x') \rightarrow
 \forall (a:A) (b:B), P a b.

End Well_founded_2.

Notation Acc_iter := Fix_F (*only parsing*). **Notation** Acc_iter_2 := Fix_F_2 (*only parsing*).

Section Acc_generator.

Variable A : Type.

Variable R : A \rightarrow A \rightarrow Prop.

Fixpoint Acc_intro_generator n (wf : well_founded R) :=

 match n with
 | O \Rightarrow wf
 | S n \Rightarrow **fun** x \Rightarrow Acc_intro x (**fun** y $_ \Rightarrow$ Acc_intro_generator n (Acc_intro_generator n
 wf) y)
 end.

End Acc_generator.

Chapter 10

Library **Coq.Init.Nat**

Require Import Notations Logic Datatypes.

Local Open Scope *nat_scope*.

10.1 Peano natural numbers, definitions of operations

This file is meant to be used as a whole module, without importing it, leading to qualified definitions (e.g. `Nat.pred`)

Definition `t` := `nat`.

10.1.1 Constants

Definition `zero` := 0.

Definition `one` := 1.

Definition `two` := 2.

10.1.2 Basic operations

Definition `succ` := `S`.

Definition `pred` `n` :=
 `match` `n` `with`
 | 0 \Rightarrow `n`
 | `S` `u` \Rightarrow `u`
 `end`.

Fixpoint `add` `n` `m` :=
 `match` `n` `with`
 | 0 \Rightarrow `m`
 | `S` `p` \Rightarrow `S` (`p` + `m`)
 `end`

where "`n` + `m`" := (`add` `n` `m`) : *nat_scope*.

Definition double $n := n + n$.

Fixpoint mul $n\ m :=$
 match n with
 | 0 \Rightarrow 0
 | S $p \Rightarrow m + p \times m$
 end

where "n * m" := (mul $n\ m$) : nat_scope.

Truncated subtraction: $n-m$ is 0 if $n \leq m$

Fixpoint sub $n\ m :=$
 match n, m with
 | S $k, S\ l \Rightarrow k - l$
 | -, - $\Rightarrow n$
 end

where "n - m" := (sub $n\ m$) : nat_scope.

10.1.3 Comparisons

Fixpoint eqb $n\ m : \text{bool} :=$
 match n, m with
 | 0, 0 \Rightarrow true
 | 0, S _ \Rightarrow false
 | S _, 0 \Rightarrow false
 | S $n', S\ m' \Rightarrow \text{eqb } n'\ m'$
 end.

Fixpoint leb $n\ m : \text{bool} :=$
 match n, m with
 | 0, _ \Rightarrow true
 | _, 0 \Rightarrow false
 | S $n', S\ m' \Rightarrow \text{leb } n'\ m'$
 end.

Definition ltb $n\ m := \text{leb } (S\ n)\ m$.

Infix "==" := eqb (at level 70) : nat_scope.

Infix "<=" := leb (at level 70) : nat_scope.

Infix "<" := ltb (at level 70) : nat_scope.

Fixpoint compare $n\ m : \text{comparison} :=$
 match n, m with
 | 0, 0 \Rightarrow Eq
 | 0, S _ \Rightarrow Lt
 | S _, 0 \Rightarrow Gt
 | S $n', S\ m' \Rightarrow \text{compare } n'\ m'$
 end.

Infix "?=" := compare (at level 70) : nat_scope.

10.1.4 Minimum, maximum

```
Fixpoint max n m :=
  match n, m with
  | 0, _ => m
  | S n', 0 => n
  | S n', S m' => S (max n' m')
  end.
```

```
Fixpoint min n m :=
  match n, m with
  | 0, _ => 0
  | S n', 0 => 0
  | S n', S m' => S (min n' m')
  end.
```

10.1.5 Parity tests

```
Fixpoint even n : bool :=
  match n with
  | 0 => true
  | 1 => false
  | S (S n') => even n'
  end.
```

Definition odd n := negb (even n).

10.1.6 Power

```
Fixpoint pow n m :=
  match m with
  | 0 => 1
  | S m => n × (n^m)
  end
```

where "n ^ m" := (pow n m) : nat_scope.

10.1.7 Euclidean division

This division is linear and tail-recursive. In *divmod*, *y* is the predecessor of the actual divisor, and *u* is *y* minus the real remainder

```
Fixpoint divmod x y q u :=
  match x with
  | 0 => (q, u)
  | S x' => match u with
    | 0 => divmod x' y (S q) y
    | S u' => divmod x' y q u'
```

```

        end
    end.

Definition div x y :=
  match y with
  | 0 => y
  | S y' => fst (divmod x y' 0 y')
  end.

Definition modulo x y :=
  match y with
  | 0 => y
  | S y' => y' - snd (divmod x y' 0 y')
  end.

Infix "/" := div : nat_scope.
Infix "mod" := modulo (at level 40, no associativity) : nat_scope.

```

10.1.8 Greatest common divisor

We use Euclid algorithm, which is normally not structural, but Coq is now clever enough to accept this (behind modulo there is a subtraction, which now preserves being a subterm)

```

Fixpoint gcd a b :=
  match a with
  | 0 => b
  | S a' => gcd (b mod (S a')) (S a')
  end.

```

10.1.9 Square

```

Definition square n := n × n.

```

10.1.10 Square root

The following square root function is linear (and tail-recursive). With Peano representation, we can't do better. For faster algorithm, see Psqrt/Zsqrt/Nsqrt...

We search the square root of $n = k + p^2 + (q - r)$ with $q = 2p$ and $0 \leq r \leq q$. We start with $p=q=r=0$, hence looking for the square root of $n = k$. Then we progressively decrease k and r . When $k = S k'$ and $r=0$, it means we can use $(S p)$ as new sqrt candidate, since $(S k') + p^2 + 2p = k' + (S p)^2$. When k reaches 0, we have found the biggest p^2 square contained in n , hence the square root of n is p .

```

Fixpoint sqrt_iter k p q r :=
  match k with
  | 0 => p
  | S k' => match r with
    | 0 => sqrt_iter k' (S p) (S (S q)) (S (S q))
    | S r' => sqrt_iter k' p q r'
  end
end

```

end.

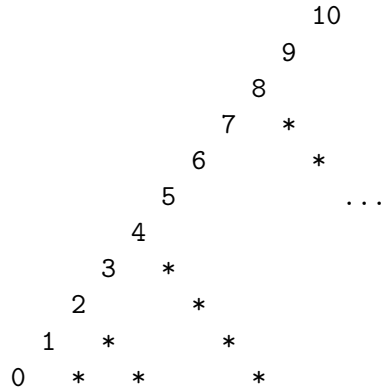
Definition `sqrt` $n := \text{sqrt_iter } n \ 0 \ 0 \ 0$.

10.1.11 Log2

This base-2 logarithm is linear and tail-recursive.

In `log2_iter`, we maintain the logarithm p of the counter q , while r is the distance between q and the next power of 2, more precisely $q + S \ r = 2^{(S \ p)}$ and $r < 2^p$. At each recursive call, q goes up while r goes down. When r is 0, we know that q has almost reached a power of 2, and we increase p at the next call, while resetting r to q .

Graphically (numbers are q , stars are r) :



We stop when k , the global downward counter reaches 0. At that moment, q is the number we're considering (since $k+q$ is invariant), and p its logarithm.

```

Fixpoint log2_iter  $k \ p \ q \ r :=$ 
  match  $k$  with
  | 0  $\Rightarrow p$ 
  | S  $k' \Rightarrow$  match  $r$  with
    | 0  $\Rightarrow$  log2_iter  $k' \ (S \ p) \ (S \ q) \ q$ 
    | S  $r' \Rightarrow$  log2_iter  $k' \ p \ (S \ q) \ r'$ 
  end
end.
```

Definition `log2` $n := \text{log2_iter } (\text{pred } n) \ 0 \ 1 \ 0$.

Iterator on natural numbers

Definition `iter` $(n:\text{nat}) \ \{A\} \ (f:A \rightarrow A) \ (x:A) : A :=$
`nat_rect (fun _ $\Rightarrow A$) $x \ (\text{fun } _ \Rightarrow f) \ n$.`

Bitwise operations

We provide here some bitwise operations for unary numbers. Some might be really naive, they are just there for fulfilling the same interface as other for natural representations. As soon as binary representations such as NArith are available, it is clearly better to convert to/from them and use their ops.

Fixpoint `div2` $n :=$

```

match n with
| 0 ⇒ 0
| S 0 ⇒ 0
| S (S n') ⇒ S (div2 n')
end.

Fixpoint testbit a n : bool :=
match n with
| 0 ⇒ odd a
| S n ⇒ testbit (div2 a) n
end.

Definition shiftl a := nat_rect _ a (fun _ ⇒ double).
Definition shiftr a := nat_rect _ a (fun _ ⇒ div2).

Fixpoint bitwise (op:bool→bool→bool) n a b :=
match n with
| 0 ⇒ 0
| S n' ⇒
  (if op (odd a) (odd b) then 1 else 0) +
  2*(bitwise op n' (div2 a) (div2 b))
end.

Definition land a b := bitwise andb a a b.
Definition lor a b := bitwise orb (max a b) a b.
Definition ldiff a b := bitwise (fun b b' ⇒ andb b (negb b')) a a b.
Definition lxor a b := bitwise xorb (max a b) a b.

```

Chapter 11

Library **Coq.Logic.Berardi**

This file formalizes Berardi's paradox which says that in the calculus of constructions, excluded middle (EM) and axiom of choice (AC) imply proof irrelevance (PI). Here, the axiom of choice is not necessary because of the use of inductive types.

```
@article{Barbanera-Berardi:JFP96,  
  author    = {F. Barbanera and S. Berardi},  
  title     = {Proof-irrelevance out of Excluded-middle and Choice  
              in the Calculus of Constructions},  
  journal   = {Journal of Functional Programming},  
  year      = {1996},  
  volume    = {6},  
  number    = {3},  
  pages     = {519-525}  
}
```

Set Implicit Arguments.

Section Berardis_paradox.

Excluded middle **Hypothesis** $EM : \forall P:\text{Prop}, P \vee \neg P$.

Conditional on any proposition. **Definition** $\text{IFProp} (P B:\text{Prop}) (e1 e2:P) :=$

```
match EM B with  
| or_introl _  $\Rightarrow e1$   
| or_intror _  $\Rightarrow e2$   
end.
```

Axiom of choice applied to disjunction. Provable in Coq because of dependent elimination.

Lemma $\text{AC_IF} :$

```
 $\forall (P B:\text{Prop}) (e1 e2:P) (Q:P \rightarrow \text{Prop}),$   
 $(B \rightarrow Q e1) \rightarrow (\neg B \rightarrow Q e2) \rightarrow Q (\text{IFProp } B e1 e2).$ 
```

We assume a type with two elements. They play the role of booleans. The main theorem under the current assumptions is that $T=F$ **Variable** $\text{Bool} : \text{Prop}$.

Variable $T : \text{Bool}$.

Variable $F : \text{Bool}$.

The powerset operator **Definition** $\text{pow} (P:\text{Prop}) := P \rightarrow \text{Bool}$.

A piece of theory about retracts **Section** `Retracts`.

Variables $A B : \text{Prop}$.

Record `retract` : $\text{Prop} :=$

$\{i : A \rightarrow B; j : B \rightarrow A; \text{inv} : \forall a:A, j (i a) = a\}$.

Record `retract_cond` : $\text{Prop} :=$

$\{i2 : A \rightarrow B; j2 : B \rightarrow A; \text{inv2} : \text{retract} \rightarrow \forall a:A, j2 (i2 a) = a\}$.

The dependent elimination above implies the axiom of choice:

Lemma `AC` : $\forall r:\text{retract_cond}, \text{retract} \rightarrow \forall a:A, r.(j2) (r.(i2) a) = a$.

End `Retracts`.

This lemma is basically a commutation of implication and existential quantification: $(\exists x \mid A \rightarrow P(x)) \Leftrightarrow (A \rightarrow \exists x \mid P(x))$ which is provable in classical logic (\Rightarrow is already provable in intuitionistic logic).

Lemma `L1` : $\forall A B:\text{Prop}, \text{retract_cond} (\text{pow } A) (\text{pow } B)$.

The paradoxical set **Definition** $U := \forall P:\text{Prop}, \text{pow } P$.

Bijection between U and $(\text{pow } U)$ **Definition** $f (u:U) : \text{pow } U := u \ U$.

Definition $g (h:\text{pow } U) : U :=$

$\text{fun } X \Rightarrow \text{let } lX := j2 (L1 X U) \text{ in let } rU := i2 (L1 U U) \text{ in } lX (rU h)$.

We deduce that the powerset of U is a retract of U . This lemma is stated in Berardi's article, but is not used afterwards. **Lemma** `retract_pow_U_U` : $\text{retract} (\text{pow } U) U$.

Encoding of Russel's paradox

The boolean negation. **Definition** `Not_b` $(b:\text{Bool}) := \text{IFProp} (b = T) F T$.

the set of elements not belonging to itself **Definition** `R` : $U := g (\text{fun } u:U \Rightarrow \text{Not_b } (u \ U u))$.

Lemma `not_has_fixpoint` : $R \ R = \text{Not_b} (R \ R)$.

Theorem `classical_proof_irrelevance` : $T = F$.

End `Berardis_paradox`.

Chapter 12

Library `Coq.Logic.ChoiceFacts`

Some facts and definitions concerning choice and description in intuitionistic logic.

We investigate the relations between the following choice and description principles

- `AC_rel` = relational form of the (non extensional) axiom of choice (a “set-theoretic” axiom of choice)
- `AC_fun` = functional form of the (non extensional) axiom of choice (a “type-theoretic” axiom of choice)
- `DC_fun` = functional form of the dependent axiom of choice
- `ACw_fun` = functional form of the countable axiom of choice
- `AC!` = functional relation reification (known as axiom of unique choice in topos theory, sometimes called principle of definite description in the context of constructive type theory)
- `GAC_rel` = guarded relational form of the (non extensional) axiom of choice
- `GAC_fun` = guarded functional form of the (non extensional) axiom of choice
- `GAC!` = guarded functional relation reification
- `OAC_rel` = “omniscient” relational form of the (non extensional) axiom of choice
- `OAC_fun` = “omniscient” functional form of the (non extensional) axiom of choice (called `AC*` in Bell [*Bell*])
- `OAC!`
- `ID_iota` = intuitionistic definite description
- `ID_epsilon` = intuitionistic indefinite description
- `D_iota` = (weakly classical) definite description principle
- `D_epsilon` = (weakly classical) indefinite description principle
- `PI` = proof irrelevance

- IGP = independence of general premises (an unconstrained generalisation of the constructive principle of independence of premises)
- Drinker = drinker's paradox (small form) (called Ex in Bell [*Bell*])

We let also

- IPL₂ = 2nd-order impredicative minimal predicate logic (with ex. quant.)
- IPL² = 2nd-order functional minimal predicate logic (with ex. quant.)
- IPL₂² = 2nd-order impredicative, 2nd-order functional minimal pred. logic (with ex. quant.)

with no prerequisite on the non-emptiness of domains

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3.1. typed IPL₂ + Sigma-types + PI |- AC_{rel} = GAC_{rel} and IPL₂ |- AC_{rel} + IGP -> GAC_{rel} and IPL₂ |- GAC_{rel} = OAC_{rel}

3.2. IPL² |- AC_{fun} + IGP = GAC_{fun} = OAC_{fun} = AC_{fun} + Drinker

3.3. D_{iota} -> ID_{iota} and D_{epsilon} <-> ID_{epsilon} + Drinker

4. Derivability of choice for decidable relations with well-ordered codomain

5. Equivalence of choices on dependent or non dependent functional types

6. Non contradiction of constructive descriptions wrt functional choices

7. Definite description transports classical logic to the computational world

8. Choice -> Dependent choice -> Countable choice

References:

[*Bell*] John L. Bell, Choice principles in intuitionistic set theory, unpublished.

[*Bell93*] John L. Bell, Hilbert's Epsilon Operator in Intuitionistic Type Theories, Mathematical Logic Quarterly, volume 39, 1993.

[*Carlström05*] Jesper Carlström, Interpreting descriptions in intentional type theory, Journal of Symbolic Logic 70(2):488-514, 2005.

Set Implicit Arguments.

12.1 Definitions

Choice, reification and description schemes

We make them all polymorphic. Most of them have existentials as conclusion so they require polymorphism otherwise their first application (e.g. to an existential in **Set**) will fix the level of *A*.

Section ChoiceSchemes.

Variables *A B* :Type.

Variable *P* :*A*→Prop.

Variable *R* :*A*→*B*→Prop.

12.1.1 Constructive choice and description

AC_rel

Definition RelationalChoice_on :=

$$\begin{aligned} & \forall R:A \rightarrow B \rightarrow \text{Prop}, \\ & (\forall x:A, \exists y:B, R\ x\ y) \rightarrow \\ & (\exists R':A \rightarrow B \rightarrow \text{Prop}, \text{subrelation } R' \ R \wedge \forall x, \exists! y, R'\ x\ y). \end{aligned}$$

AC_fun

Definition FunctionalChoice_on :=

$$\begin{aligned} & \forall R:A \rightarrow B \rightarrow \text{Prop}, \\ & (\forall x:A, \exists y:B, R\ x\ y) \rightarrow \\ & (\exists f:A \rightarrow B, \forall x:A, R\ x\ (f\ x)). \end{aligned}$$

DC_fun

Definition FunctionalDependentChoice_on :=

$$\begin{aligned} & \forall (R:A \rightarrow A \rightarrow \text{Prop}), \\ & (\forall x, \exists y, R\ x\ y) \rightarrow \forall x0, \\ & (\exists f:\text{nat} \rightarrow A, f\ 0 = x0 \wedge \forall n, R\ (f\ n)\ (f\ (\text{S } n))). \end{aligned}$$

ACw_fun

Definition FunctionalCountableChoice_on :=

$$\begin{aligned} & \forall (R:\text{nat} \rightarrow A \rightarrow \text{Prop}), \\ & (\forall n, \exists y, R\ n\ y) \rightarrow \\ & (\exists f:\text{nat} \rightarrow A, \forall n, R\ n\ (f\ n)). \end{aligned}$$

AC! or Functional Relation Reification (known as Axiom of Unique Choice in topos theory; also called principle of definite description)

Definition FunctionalRelReification_on :=

$$\begin{aligned} & \forall R:A \rightarrow B \rightarrow \text{Prop}, \\ & (\forall x:A, \exists! y:B, R\ x\ y) \rightarrow \\ & (\exists f:A \rightarrow B, \forall x:A, R\ x\ (f\ x)). \end{aligned}$$

ID_epsilon (constructive version of indefinite description; combined with proof-irrelevance, it may be connected to Carlström's type theory with a constructive indefinite description operator)

Definition ConstructiveIndefiniteDescription_on :=

$$\begin{aligned} & \forall P:A \rightarrow \text{Prop}, \\ & (\exists x, P\ x) \rightarrow \{x:A \mid P\ x\}. \end{aligned}$$

ID_iota (constructive version of definite description; combined with proof-irrelevance, it may be connected to Carlström's and Stenlund's type theory with a constructive definite description operator)

Definition ConstructiveDefiniteDescription_on :=

$$\begin{aligned} & \forall P:A \rightarrow \text{Prop}, \\ & (\exists! x, P\ x) \rightarrow \{x:A \mid P\ x\}. \end{aligned}$$

12.1.2 Weakly classical choice and description

GAC_rel

Definition GuardedRelationalChoice_on :=
 $\forall P : A \rightarrow \text{Prop}, \forall R : A \rightarrow B \rightarrow \text{Prop},$
 $(\forall x : A, P x \rightarrow \exists y : B, R x y) \rightarrow$
 $(\exists R' : A \rightarrow B \rightarrow \text{Prop},$
 $\text{subrelation } R' R \wedge \forall x, P x \rightarrow \exists! y, R' x y).$
GAC_fun

Definition GuardedFunctionalChoice_on :=
 $\forall P : A \rightarrow \text{Prop}, \forall R : A \rightarrow B \rightarrow \text{Prop},$
 $\text{inhabited } B \rightarrow$
 $(\forall x : A, P x \rightarrow \exists y : B, R x y) \rightarrow$
 $(\exists f : A \rightarrow B, \forall x, P x \rightarrow R x (f x)).$
GFR_fun

Definition GuardedFunctionalRelReification_on :=
 $\forall P : A \rightarrow \text{Prop}, \forall R : A \rightarrow B \rightarrow \text{Prop},$
 $\text{inhabited } B \rightarrow$
 $(\forall x : A, P x \rightarrow \exists! y : B, R x y) \rightarrow$
 $(\exists f : A \rightarrow B, \forall x : A, P x \rightarrow R x (f x)).$
OAC_rel

Definition OmniscientRelationalChoice_on :=
 $\forall R : A \rightarrow B \rightarrow \text{Prop},$
 $\exists R' : A \rightarrow B \rightarrow \text{Prop},$
 $\text{subrelation } R' R \wedge \forall x : A, (\exists y : B, R x y) \rightarrow \exists! y, R' x y.$
OAC_fun

Definition OmniscientFunctionalChoice_on :=
 $\forall R : A \rightarrow B \rightarrow \text{Prop},$
 $\text{inhabited } B \rightarrow$
 $\exists f : A \rightarrow B, \forall x : A, (\exists y : B, R x y) \rightarrow R x (f x).$
D_epsilon

Definition EpsilonStatement_on :=
 $\forall P : A \rightarrow \text{Prop},$
 $\text{inhabited } A \rightarrow \{ x : A \mid (\exists x, P x) \rightarrow P x \}.$
D_iota

Definition IotaStatement_on :=
 $\forall P : A \rightarrow \text{Prop},$
 $\text{inhabited } A \rightarrow \{ x : A \mid (\exists! x, P x) \rightarrow P x \}.$

End ChoiceSchemes.

Generalized schemes

Notation RelationalChoice :=
 $(\forall A B : \text{Type}, \text{RelationalChoice_on } A B).$

Notation FunctionalChoice :=
 $(\forall A B : \text{Type}, \text{FunctionalChoice_on } A B).$

Definition FunctionalDependentChoice :=
 $(\forall A : \text{Type}, \text{FunctionalDependentChoice_on } A).$
Definition FunctionalCountableChoice :=
 $(\forall A : \text{Type}, \text{FunctionalCountableChoice_on } A).$
Notation FunctionalChoiceOnInhabitedSet :=
 $(\forall A B : \text{Type}, \text{inhabited } B \rightarrow \text{FunctionalChoice_on } A B).$
Notation FunctionalRelReification :=
 $(\forall A B : \text{Type}, \text{FunctionalRelReification_on } A B).$
Notation GuardedRelationalChoice :=
 $(\forall A B : \text{Type}, \text{GuardedRelationalChoice_on } A B).$
Notation GuardedFunctionalChoice :=
 $(\forall A B : \text{Type}, \text{GuardedFunctionalChoice_on } A B).$
Notation GuardedFunctionalRelReification :=
 $(\forall A B : \text{Type}, \text{GuardedFunctionalRelReification_on } A B).$
Notation OmniscientRelationalChoice :=
 $(\forall A B : \text{Type}, \text{OmniscientRelationalChoice_on } A B).$
Notation OmniscientFunctionalChoice :=
 $(\forall A B : \text{Type}, \text{OmniscientFunctionalChoice_on } A B).$
Notation ConstructiveDefiniteDescription :=
 $(\forall A : \text{Type}, \text{ConstructiveDefiniteDescription_on } A).$
Notation ConstructiveIndefiniteDescription :=
 $(\forall A : \text{Type}, \text{ConstructiveIndefiniteDescription_on } A).$
Notation IotaStatement :=
 $(\forall A : \text{Type}, \text{IotaStatement_on } A).$
Notation EpsilonStatement :=
 $(\forall A : \text{Type}, \text{EpsilonStatement_on } A).$
 Subclassical schemes
Definition ProofIrrelevance :=
 $\forall (A:\text{Prop}) (a1\ a2:A), a1 = a2.$
Definition IndependenceOfGeneralPremises :=
 $\forall (A:\text{Type}) (P:A \rightarrow \text{Prop}) (Q:\text{Prop}),$
 $\text{inhabited } A \rightarrow$
 $(Q \rightarrow \exists x, P\ x) \rightarrow \exists x, Q \rightarrow P\ x.$
Definition SmallDrinker'sParadox :=
 $\forall (A:\text{Type}) (P:A \rightarrow \text{Prop}), \text{inhabited } A \rightarrow$
 $\exists x, (\exists x, P\ x) \rightarrow P\ x.$

12.2 AC_{rel} + AC! = AC_{fun}

We show that the functional formulation of the axiom of Choice (usual formulation in type theory) is equivalent to its relational formulation (only formulation of set theory) + functional relation reification (aka axiom of unique choice, or, principle of (parametric) definite descriptions)

This shows that the axiom of choice can be assumed (under its relational formulation) without known inconsistency with classical logic, though functional relation reification conflicts with classical logic

Lemma `description_rel_choice_imp_func_choice` :

$\forall A B : \text{Type},$
 $\text{FunctionalRelReification_on } A B \rightarrow \text{RelationalChoice_on } A B \rightarrow \text{FunctionalChoice_on } A B.$

Lemma `func_choice_imp_rel_choice` :

$\forall A B : \text{Type}, \text{FunctionalChoice_on } A B \rightarrow \text{RelationalChoice_on } A B.$

Lemma `func_choice_imp_description` :

$\forall A B : \text{Type}, \text{FunctionalChoice_on } A B \rightarrow \text{FunctionalRelReification_on } A B.$

Corollary `FunChoice_Equiv_RelChoice_and_ParamDefinDescr` :

$\forall A B : \text{Type}, \text{FunctionalChoice_on } A B \leftrightarrow$
 $\text{RelationalChoice_on } A B \wedge \text{FunctionalRelReification_on } A B.$

12.3 Connection between the guarded, non guarded and omniscient choices

We show that the guarded formulations of the axiom of choice are equivalent to their “omniscient” variant and comes from the non guarded formulation in presence either of the independence of general premises or subset types (themselves derivable from subtypes thanks to proof- irrelevance)

12.3.1 $\text{AC_rel} + \text{PI} \rightarrow \text{GAC_rel}$ and $\text{AC_rel} + \text{IGP} \rightarrow \text{GAC_rel}$ and $\text{GAC_rel} = \text{OAC_rel}$

Lemma `rel_choice_and_proof_irrel_imp_guarded_rel_choice` :

$\text{RelationalChoice} \rightarrow \text{ProofIrrelevance} \rightarrow \text{GuardedRelationalChoice}.$

Lemma `rel_choice_indep_of_general_premises_imp_guarded_rel_choice` :

$\forall A B : \text{Type}, \text{inhabited } B \rightarrow \text{RelationalChoice_on } A B \rightarrow$
 $\text{IndependenceOfGeneralPremises} \rightarrow \text{GuardedRelationalChoice_on } A B.$

Lemma `guarded_rel_choice_imp_rel_choice` :

$\forall A B : \text{Type}, \text{GuardedRelationalChoice_on } A B \rightarrow \text{RelationalChoice_on } A B.$

Lemma `subset_types_imp_guarded_rel_choice_iff_rel_choice` :

$\text{ProofIrrelevance} \rightarrow (\text{GuardedRelationalChoice} \leftrightarrow \text{RelationalChoice}).$

$\text{OAC_rel} = \text{GAC_rel}$

Corollary `guarded_iff_omniscient_rel_choice` :

$\text{GuardedRelationalChoice} \leftrightarrow \text{OmniscientRelationalChoice}.$

12.3.2 $\text{AC_fun} + \text{IGP} = \text{GAC_fun} = \text{OAC_fun} = \text{AC_fun} + \text{Drinker}$

$\text{AC_fun} + \text{IGP} = \text{GAC_fun}$

Lemma `guarded_fun_choice_imp_indep_of_general_premises` :

$\text{GuardedFunctionalChoice} \rightarrow \text{IndependenceOfGeneralPremises}.$

Lemma guarded_fun_choice_imp_fun_choice :

GuardedFunctionalChoice \rightarrow FunctionalChoiceOnInhabitedSet.

Lemma fun_choice_and_indep_general_prem_imp_guarded_fun_choice :

FunctionalChoiceOnInhabitedSet \rightarrow IndependenceOfGeneralPremises
 \rightarrow GuardedFunctionalChoice.

Corollary fun_choice_and_indep_general_prem_iff_guarded_fun_choice :

FunctionalChoiceOnInhabitedSet \wedge IndependenceOfGeneralPremises
 \leftrightarrow GuardedFunctionalChoice.

AC_fun + Drinker = OAC_fun

This was already observed by Bell [*Bell*]

Lemma omniscient_fun_choice_imp_small_drinker :

OmniscientFunctionalChoice \rightarrow SmallDrinker'sParadox.

Lemma omniscient_fun_choice_imp_fun_choice :

OmniscientFunctionalChoice \rightarrow FunctionalChoiceOnInhabitedSet.

Lemma fun_choice_and_small_drinker_imp_omniscient_fun_choice :

FunctionalChoiceOnInhabitedSet \rightarrow SmallDrinker'sParadox
 \rightarrow OmniscientFunctionalChoice.

Corollary fun_choice_and_small_drinker_iff_omniscient_fun_choice :

FunctionalChoiceOnInhabitedSet \wedge SmallDrinker'sParadox
 \leftrightarrow OmniscientFunctionalChoice.

OAC_fun = GAC_fun

This is derivable from the intuitionistic equivalence between IGP and Drinker but we give a direct proof

Theorem guarded_iff_omniscient_fun_choice :

GuardedFunctionalChoice \leftrightarrow OmniscientFunctionalChoice.

12.3.3 D_iota \rightarrow ID_iota and D_epsilon \leftrightarrow ID_epsilon + Drinker

D_iota \rightarrow ID_iota

Lemma iota_imp_constructive_definite_description :

IotaStatement \rightarrow ConstructiveDefiniteDescription.

ID_epsilon + Drinker \leftrightarrow D_epsilon

Lemma epsilon_imp_constructive_indefinite_description:

EpsilonStatement \rightarrow ConstructiveIndefiniteDescription.

Lemma constructive_indefinite_description_and_small_drinker_imp_epsilon :

SmallDrinker'sParadox \rightarrow ConstructiveIndefiniteDescription \rightarrow
EpsilonStatement.

Lemma epsilon_imp_small_drinker :

EpsilonStatement \rightarrow SmallDrinker'sParadox.

Theorem constructive_indefinite_description_and_small_drinker_iff_epsilon :

(SmallDrinker'sParadox \times ConstructiveIndefiniteDescription \rightarrow

EpsilonStatement) ×
 (EpsilonStatement →
 SmallDrinker'sParadox × ConstructiveIndefiniteDescription).

12.4 Derivability of choice for decidable relations with well-ordered codomain

Countable codomains, such as *nat*, can be equipped with a well-order, which implies the existence of a least element on inhabited decidable subsets. As a consequence, the relational form of the axiom of choice is derivable on *nat* for decidable relations.

We show instead that functional relation reification and the functional form of the axiom of choice are equivalent on decidable relation with *nat* as codomain

Require Import Wf_nat.

Require Import Decidable.

Definition FunctionalChoice_on_rel (A B:Type) (R:A→B→Prop) :=
 (∀ x:A, ∃ y : B, R x y) →
 ∃ f : A → B, (∀ x:A, R x (f x)).

Lemma classical_denumerable_description_imp_fun_choice :

∀ A:Type,
 FunctionalRelReification_on A nat →
 ∀ R:A→nat→Prop,
 (∀ x y, decidable (R x y)) → FunctionalChoice_on_rel R.

12.5 Choice on dependent and non dependent function types are equivalent

12.5.1 Choice on dependent and non dependent function types are equivalent

Definition DependentFunctionalChoice_on (A:Type) (B:A → Type) :=
 ∀ R:∀ x:A, B x → Prop,
 (∀ x:A, ∃ y : B x, R x y) →
 (∃ f : (∀ x:A, B x), ∀ x:A, R x (f x)).

Notation DependentFunctionalChoice :=

(∀ A (B:A→Type), DependentFunctionalChoice_on B).

The easy part

Theorem dep_non_dep_functional_choice :

DependentFunctionalChoice → FunctionalChoice.

Deriving choice on product types requires some computation on singleton propositional types, so we need computational conjunction projections and dependent elimination of conjunction and equality

Scheme and_indd := Induction for and Sort Prop.

Scheme eq_indd := Induction for eq Sort Prop.

Definition `proj1_inf (A B:Prop) (p : A ∧ B) :=
 let (a,b) := p in a.`

Theorem `non_dep_dep_functional_choice :`
`FunctionalChoice → DependentFunctionalChoice.`

12.5.2 Reification of dependent and non dependent functional relation are equivalent

Definition `DependentFunctionalRelReification_on (A:Type) (B:A → Type) :=
 ∀ (R:∀ x:A, B x → Prop),
 (∀ x:A, ∃! y : B x, R x y) →
 (∃ f : (∀ x:A, B x), ∀ x:A, R x (f x)).`

Notation `DependentFunctionalRelReification :=
 (∀ A (B:A→Type), DependentFunctionalRelReification_on B).`

The easy part

Theorem `dep_non_dep_functional_rel_reification :`
`DependentFunctionalRelReification → FunctionalRelReification.`

Deriving choice on product types requires some computation on singleton propositional types, so we need computational conjunction projections and dependent elimination of conjunction and equality

Theorem `non_dep_dep_functional_rel_reification :`
`FunctionalRelReification → DependentFunctionalRelReification.`

Corollary `dep_iff_non_dep_functional_rel_reification :`
`FunctionalRelReification ↔ DependentFunctionalRelReification.`

12.6 Non contradiction of constructive descriptions wrt functional axioms of choice

12.6.1 Non contradiction of indefinite description

Lemma `relative_non_contradiction_of_indefinite_descr :`
`∀ C:Prop, (ConstructiveIndefiniteDescription → C)
 → (FunctionalChoice → C).`

Lemma `constructive_indefinite_descr_fun_choice :`
`ConstructiveIndefiniteDescription → FunctionalChoice.`

12.6.2 Non contradiction of definite description

Lemma `relative_non_contradiction_of_definite_descr :`
`∀ C:Prop, (ConstructiveDefiniteDescription → C)
 → (FunctionalRelReification → C).`

Lemma `constructive_definite_descr_fun_reification :`

`ConstructiveDefiniteDescription` \rightarrow `FunctionalRelReification`.

Remark, the following corollaries morally hold:

Definition `In_propositional_context` ($A:Type$) := forall $C:Prop$, ($A \rightarrow C$) \rightarrow C .

Corollary `constructive_definite_descr_in_prop_context_iff_fun_reification` : `In_propositional_context`
`ConstructiveIndefiniteDescription` \leftrightarrow `FunctionalChoice`.

Corollary `constructive_definite_descr_in_prop_context_iff_fun_reification` : `In_propositional_context`
`ConstructiveDefiniteDescription` \leftrightarrow `FunctionalRelReification`.

but expecting *FunctionalChoice* (resp. *FunctionalRelReification*) to be applied on the same Type universes on both sides of the first (resp. second) equivalence breaks the stratification of universes.

12.7 Excluded-middle + definite description \Rightarrow computational excluded-middle

The idea for the following proof comes from [*ChicliPottierSimpson02*]

Classical logic and axiom of unique choice (i.e. functional relation reification), as shown in [*ChicliPottierSimpson02*], implies the double-negation of excluded-middle in **Set** (which is incompatible with the impredicativity of **Set**).

We adapt the proof to show that constructive definite description transports excluded-middle from **Prop** to **Set**.

[*ChicliPottierSimpson02*] Laurent Chicli, Loïc Pottier, Carlos Simpson, Mathematical Quotients and Quotient Types in Coq, Proceedings of TYPES 2002, Lecture Notes in Computer Science 2646, Springer Verlag.

Require Import Setoid.

Theorem `constructive_definite_descr_excluded_middle` :

($\forall A : Type$, `ConstructiveDefiniteDescription_on` A) \rightarrow
($\forall P:Prop$, $P \vee \neg P$) \rightarrow ($\forall P:Prop$, $\{P\} + \{\neg P\}$).

Corollary `fun_reification_descr_computational_excluded_middle_in_prop_context` :

`FunctionalRelReification` \rightarrow
($\forall P:Prop$, $P \vee \neg P$) \rightarrow
 $\forall C:Prop$, (($\forall P:Prop$, $\{P\} + \{\neg P\}$) $\rightarrow C$) $\rightarrow C$.

12.8 Choice \Rightarrow Dependent choice \Rightarrow Countable choice

Require Import Arith.

Theorem `functional_choice_imp_functional_dependent_choice` :

`FunctionalChoice` \rightarrow `FunctionalDependentChoice`.

Theorem `functional_dependent_choice_imp_functional_countable_choice` :

`FunctionalDependentChoice` \rightarrow `FunctionalCountableChoice`.

Chapter 13

Library `Coq.Logic.ClassicalChoice`

This file provides classical logic and functional choice; this especially provides both indefinite descriptions and choice functions but this is weaker than providing epsilon operator and classical logic as the indefinite descriptions provided by the axiom of choice can be used only in a propositional context (especially, they cannot be used to build choice functions outside the scope of a theorem proof)

This file extends `ClassicalUniqueChoice.v` with full choice. As `ClassicalUniqueChoice.v`, it implies the double-negation of excluded-middle in `Set` and leads to a classical world populated with non computable functions. Especially it conflicts with the impredicativity of `Set`, knowing that *true*≠*false* in `Set`.

```
Require Export ClassicalUniqueChoice.
```

```
Require Export RelationalChoice.
```

```
Require Import ChoiceFacts.
```

```
Set Implicit Arguments.
```

```
Definition subset (U:Type) (P Q:U→Prop) : Prop := ∀ x, P x → Q x.
```

```
Theorem singleton_choice :
```

```
  ∀ (A : Type) (P : A→Prop),  
  (∃ x : A, P x) → ∃ P' : A→Prop, subset P' P ∧ ∃! x, P' x.
```

```
Theorem choice :
```

```
  ∀ (A B : Type) (R : A→B→Prop),  
  (∀ x : A, ∃ y : B, R x y) →  
  ∃ f : A→B, (∀ x : A, R x (f x)).
```

Chapter 14

Library `Coq.Logic.ClassicalDescription`

This file provides classical logic and definite description, which is equivalent to providing classical logic and Church's iota operator

Classical logic and definite descriptions implies excluded-middle in `Set` and leads to a classical world populated with non computable functions. It conflicts with the impredicativity of `Set`

`Set Implicit Arguments.`

`Require Export Classical. Require Export Description. Require Import ChoiceFacts.`

The idea for the following proof comes from *ChicliPottierSimpson02*

Theorem `excluded_middle_informative` : $\forall P:\text{Prop}, \{P\} + \{\neg P\}$.

Theorem `classical_definite_description` :

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}), \text{inhabited } A \rightarrow$
 $\{x : A \mid (\exists! x : A, P x) \rightarrow P x\}$.

Church's iota operator

Definition `iota` $(A : \text{Type}) (i:\text{inhabited } A) (P : A \rightarrow \text{Prop}) : A$
 $:= \text{proj1_sig } (\text{classical_definite_description } P i)$.

Definition `iota_spec` $(A : \text{Type}) (i:\text{inhabited } A) (P : A \rightarrow \text{Prop}) :$
 $(\exists! x:A, P x) \rightarrow P (\text{iota } i P)$
 $:= \text{proj2_sig } (\text{classical_definite_description } P i)$.

Axiom of unique “choice” (functional reification of functional relations) **Theorem** `dependent_unique_choice` :

$\forall (A:\text{Type}) (B:A \rightarrow \text{Type}) (R:\forall x:A, B x \rightarrow \text{Prop}),$
 $(\forall x:A, \exists! y : B x, R x y) \rightarrow$
 $(\exists f : (\forall x:A, B x), \forall x:A, R x (f x))$.

Theorem `unique_choice` :

$\forall (A B:\text{Type}) (R:A \rightarrow B \rightarrow \text{Prop}),$
 $(\forall x:A, \exists! y : B, R x y) \rightarrow$
 $(\exists f : A \rightarrow B, \forall x:A, R x (f x))$.

Compatibility lemmas

`Unset Implicit Arguments.`

Definition `dependent_description` := `dependent_unique_choice`.
Definition `description` := `unique_choice`.

Chapter 15

Library Coq.Logic.ClassicalEpsilon

This file provides classical logic and indefinite description under the form of Hilbert's epsilon operator

Hilbert's epsilon operator and classical logic implies excluded-middle in **Set** and leads to a classical world populated with non computable functions. It conflicts with the impredicativity of **Set**

Require Export Classical.

Require Import ChoiceFacts.

Set Implicit Arguments.

Axiom *constructive_indefinite_description* :

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}),$
 $(\exists x, P x) \rightarrow \{ x : A \mid P x \}.$

Lemma *constructive_definite_description* :

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}),$
 $(\exists! x, P x) \rightarrow \{ x : A \mid P x \}.$

Theorem *excluded_middle_informative* : $\forall P : \text{Prop}, \{P\} + \{\neg P\}.$

Theorem *classical_indefinite_description* :

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}), \text{inhabited } A \rightarrow$
 $\{ x : A \mid (\exists x, P x) \rightarrow P x \}.$

Hilbert's epsilon operator

Definition *epsilon* ($A : \text{Type}$) ($i : \text{inhabited } A$) ($P : A \rightarrow \text{Prop}$) : A
:= *proj1_sig* (*classical_indefinite_description* P i).

Definition *epsilon_spec* ($A : \text{Type}$) ($i : \text{inhabited } A$) ($P : A \rightarrow \text{Prop}$) :
 $(\exists x, P x) \rightarrow P$ (*epsilon* i P)
:= *proj2_sig* (*classical_indefinite_description* P i).

Open question: is *classical_indefinite_description* constructively provable from *relational_choice* and *constructive_definite_description* (at least, using the fact that *functional_choice* is provable from *relational_choice* and *unique_choice*, we know that the double negation of *classical_indefinite_description* is provable (see *relative_non_contradiction_of_indefinite_desc*).

A proof that if P is inhabited, *epsilon* a P does not depend on the actual proof that the domain of P is inhabited (proof idea kindly provided by Pierre Cast  ran)

Lemma `epsilon_inh_irrelevance` :
 $\forall (A:\text{Type}) (i\ j : \text{inhabited } A) (P:A\rightarrow\text{Prop}),$
 $(\exists x, P\ x) \rightarrow \text{epsilon } i\ P = \text{epsilon } j\ P.$
Opaque `epsilon`.

Weaker lemmas (compatibility lemmas)

Theorem `choice` :
 $\forall (A\ B : \text{Type}) (R : A\rightarrow B\rightarrow\text{Prop}),$
 $(\forall x : A, \exists y : B, R\ x\ y) \rightarrow$
 $(\exists f : A\rightarrow B, \forall x : A, R\ x\ (f\ x)).$

Chapter 16

Library `Coq.Logic.ClassicalFacts`

Some facts and definitions about classical logic

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16.1 Prop degeneracy = excluded-middle + prop extensionality

i.e. $(\forall A, A = \text{True} \vee A = \text{False}) \leftrightarrow (\forall A, A \vee \neg A) \wedge (\forall A B, (A \leftrightarrow B) \rightarrow A = B)$

prop_degeneracy (also referred to as propositional completeness) asserts (up to consistency) that there are only two distinct formulas **Definition** `prop_degeneracy` := $\forall A:\text{Prop}, A = \text{True} \vee A = \text{False}$.

prop_extensionality asserts that equivalent formulas are equal **Definition** `prop_extensionality` := $\forall A B:\text{Prop}, (A \leftrightarrow B) \rightarrow A = B$.

excluded_middle asserts that we can reason by case on the truth or falsity of any formula **Definition** `excluded_middle` := $\forall A:\text{Prop}, A \vee \neg A$.

We show *prop_degeneracy* \leftrightarrow (*prop_extensionality* \wedge *excluded_middle*)

Lemma `prop-degen-ext` : *prop_degeneracy* \rightarrow *prop_extensionality*.

Lemma `prop-degen-em` : *prop_degeneracy* \rightarrow *excluded_middle*.

Lemma `prop_ext_em_degen` :

prop_extensionality \rightarrow *excluded_middle* \rightarrow *prop_degeneracy*.

A weakest form of propositional extensionality: extensionality for provable propositions only

Definition `provable_prop_extensionality` := $\forall A:\text{Prop}, A \rightarrow A = \text{True}$.

Lemma `provable_prop_ext` :
`prop_extensionality` \rightarrow `provable_prop_extensionality`.

16.2 Classical logic and proof-irrelevance

16.2.1 CC |- `prop_ext` + `A inhabited` \rightarrow (`A = A \rightarrow A`) \rightarrow `A` has fixpoint

We successively show that:

`prop_extensionality` implies equality of A and $A \rightarrow A$ for inhabited A , which implies the existence of a (trivial) retract from $A \rightarrow A$ to A (just take the identity), which implies the existence of a fixpoint operator in A (e.g. take the Y combinator of lambda-calculus)

Lemma `prop_ext_A_eq_A_imp_A` :
`prop_extensionality` $\rightarrow \forall A:\text{Prop}, \text{inhabited } A \rightarrow (A \rightarrow A) = A$.

Record `retract` ($A B:\text{Prop}$) : `Prop` :=
 $\{f1 : A \rightarrow B; f2 : B \rightarrow A; f1_o_f2 : \forall x:B, f1 (f2 x) = x\}$.

Lemma `prop_ext_retract_A_A_imp_A` :
`prop_extensionality` $\rightarrow \forall A:\text{Prop}, \text{inhabited } A \rightarrow \text{retract } A (A \rightarrow A)$.

Record `has_fixpoint` ($A:\text{Prop}$) : `Prop` :=
 $\{F : (A \rightarrow A) \rightarrow A; \text{Fix} : \forall f:A \rightarrow A, F f = f (F f)\}$.

Lemma `ext_prop_fixpoint` :
`prop_extensionality` $\rightarrow \forall A:\text{Prop}, \text{inhabited } A \rightarrow \text{has_fixpoint } A$.

Remark: `prop_extensionality` can be replaced in lemma `ext_prop_fixpoint` by the weakest property `provable_prop_extensionality`.

16.2.2 CC |- `prop_ext` /\ `dep elim on bool` \rightarrow `proof-irrelevance`

`proof_irrelevance` asserts equality of all proofs of a given formula **Definition** `proof_irrelevance` :=
 $\forall (A:\text{Prop}) (a1 a2:A), a1 = a2$.

Assume that we have booleans with the property that there is at most 2 booleans (which is equivalent to dependent case analysis). Consider the fixpoint of the negation function: it is either true or false by dependent case analysis, but also the opposite by fixpoint. Hence proof-irrelevance.

We then map equality of boolean proofs to proof irrelevance in all propositions.

Section `Proof_irrelevance_gen`.

Variable `bool` : `Prop`.

Variable `true` : `bool`.

Variable `false` : `bool`.

Hypothesis `bool_elim` : $\forall C:\text{Prop}, C \rightarrow C \rightarrow \text{bool} \rightarrow C$.

Hypothesis

`bool_elim_redl` : $\forall (C:\text{Prop}) (c1 c2:C), c1 = \text{bool_elim } C c1 c2 \text{ true}$.

Hypothesis

`bool_elim_redr` : $\forall (C:\text{Prop}) (c1 c2:C), c2 = \text{bool_elim } C c1 c2 \text{ false}$.

Let `bool_dep_induction` :=


```

 $\forall P:\text{bool} \rightarrow \text{Prop}, P \text{ true} \rightarrow P \text{ false} \rightarrow \forall b:\text{bool}, P b.$ 
Lemma aux : prop_extensionality  $\rightarrow$  bool_dep_induction  $\rightarrow$  true = false.
Lemma ext_prop_dep_proof_irrel_gen :
  prop_extensionality  $\rightarrow$  bool_dep_induction  $\rightarrow$  proof_irrelevance.
End Proof_irrelevance_gen.

```

In the pure Calculus of Constructions, we can define the boolean proposition $\text{bool} = (\text{C:Prop})\text{C} \rightarrow \text{C} \rightarrow \text{C}$ but we cannot prove that it has at most 2 elements.

Section Proof_irrelevance_Prop_Ext_CC.

```

Definition BoolP :=  $\forall C:\text{Prop}, C \rightarrow C \rightarrow C.$ 
Definition TrueP : BoolP := fun C c1 c2  $\Rightarrow$  c1.
Definition FalseP : BoolP := fun C c1 c2  $\Rightarrow$  c2.
Definition BoolP_elim C c1 c2 (b:BoolP) := b C c1 c2.
Definition BoolP_elim_redl (C:Prop) (c1 c2:C) :
  c1 = BoolP_elim C c1 c2 TrueP := eq_refl c1.
Definition BoolP_elim_redr (C:Prop) (c1 c2:C) :
  c2 = BoolP_elim C c1 c2 FalseP := eq_refl c2.
Definition BoolP_dep_induction :=
   $\forall P:\text{BoolP} \rightarrow \text{Prop}, P \text{ TrueP} \rightarrow P \text{ FalseP} \rightarrow \forall b:\text{BoolP}, P b.$ 
Lemma ext_prop_dep_proof_irrel_cc :
  prop_extensionality  $\rightarrow$  BoolP_dep_induction  $\rightarrow$  proof_irrelevance.
End Proof_irrelevance_Prop_Ext_CC.

```

Remark: *prop_extensionality* can be replaced in lemma *ext_prop_dep_proof_irrel_gen* by the weakest property *provable_prop_extensionality*.

16.2.3 CIC |- prop. ext. \rightarrow proof-irrelevance

In the Calculus of Inductive Constructions, inductively defined booleans enjoy dependent case analysis, hence directly proof-irrelevance from propositional extensionality.

Section Proof_irrelevance_CIC.

```

Inductive boolP : Prop :=
| trueP : boolP
| falseP : boolP.
Definition boolP_elim_redl (C:Prop) (c1 c2:C) :
  c1 = boolP_ind C c1 c2 trueP := eq_refl c1.
Definition boolP_elim_redr (C:Prop) (c1 c2:C) :
  c2 = boolP_ind C c1 c2 falseP := eq_refl c2.
Scheme boolP_indd := Induction for boolP Sort Prop.
Lemma ext_prop_dep_proof_irrel_cic : prop_extensionality  $\rightarrow$  proof_irrelevance.
End Proof_irrelevance_CIC.

```

Can we state proof irrelevance from propositional degeneracy (i.e. propositional extensionality + excluded middle) without dependent case analysis ?

Berardi [Berardi90] built a model of CC interpreting inhabited types by the set of all untyped lambda-terms. This model satisfies propositional degeneracy without satisfying proof-irrelevance (nor dependent case analysis). This implies that the previous results cannot be refined.

[Berardi90] Stefano Berardi, “Type dependence and constructive mathematics”, Ph. D. thesis, Dipartimento Matematica, Università di Torino, 1990.

16.2.4 CC |- excluded-middle + dep elim on bool -> proof-irrelevance

This is a proof in the pure Calculus of Construction that classical logic in **Prop** + dependent elimination of disjunction entails proof-irrelevance.

Reference:

[Coquand90] T. Coquand, “Metamathematical Investigations of a Calculus of Constructions”, Proceedings of Logic in Computer Science (LICS’90), 1990.

Proof skeleton: classical logic + dependent elimination of disjunction + discrimination of proofs implies the existence of a retract from **Prop** into *bool*, hence inconsistency by encoding any paradox of system U- (e.g. Hurkens’ paradox).

Require Import Hurkens.

Section Proof_irrelevance_EM_CC.

Variable *or* : **Prop** → **Prop** → **Prop**.

Variable *or_introl* : ∀ *A B*:**Prop**, *A* → *or A B*.

Variable *or_intror* : ∀ *A B*:**Prop**, *B* → *or A B*.

Hypothesis *or_elim* : ∀ *A B C*:**Prop**, (*A* → *C*) → (*B* → *C*) → *or A B* → *C*.

Hypothesis

or_elim_redl :

∀ (*A B C*:**Prop**) (*f*:*A* → *C*) (*g*:*B* → *C*) (*a*:*A*),
f a = *or_elim A B C f g (or_introl A B a)*.

Hypothesis

or_elim_redr :

∀ (*A B C*:**Prop**) (*f*:*A* → *C*) (*g*:*B* → *C*) (*b*:*B*),
g b = *or_elim A B C f g (or_intror A B b)*.

Hypothesis

or_dep_elim :

∀ (*A B*:**Prop**) (*P*:*or A B* → **Prop**),
 (∀ *a*:*A*, *P (or_introl A B a)*) →
 (∀ *b*:*B*, *P (or_intror A B b)*) → ∀ *b*:*or A B*, *P b*.

Hypothesis *em* : ∀ *A*:**Prop**, *or A* (¬ *A*).

Variable *B* : **Prop**.

Variables *b1 b2* : *B*.

p2b and *b2p* form a retract if ¬*b1*=*b2*

Let *p2b A* := *or_elim A* (¬ *A*) *B* (**fun** _ ⇒ *b1*) (**fun** _ ⇒ *b2*) (*em A*).

Let *b2p b* := *b1* = *b*.

Lemma *p2p1* : ∀ *A*:**Prop**, *A* → *b2p (p2b A)*.

Lemma *p2p2* : *b1* ≠ *b2* → ∀ *A*:**Prop**, *b2p (p2b A)* → *A*.

Using excluded-middle a second time, we get proof-irrelevance

Theorem `proof_irrelevance_cc` : $b1 = b2$.

End `Proof_irrelevance_EM_CC`.

Hurkens' paradox still holds with a retract from the *negative* fragment of **Prop** into *bool*, hence weak classical logic, i.e. $\forall A, \neg A \setminus \sim \sim A$, is enough for deriving a weak version of proof-irrelevance. This is enough to derive a contradiction from a **Set**-bound weak excluded middle with an impredicative **Set** universe.

Section `Proof_irrelevance_WEM_CC`.

Variable `or` : **Prop** \rightarrow **Prop** \rightarrow **Prop**.

Variable `or_introl` : $\forall A B:\text{Prop}, A \rightarrow \text{or } A B$.

Variable `or_intror` : $\forall A B:\text{Prop}, B \rightarrow \text{or } A B$.

Hypothesis `or_elim` : $\forall A B C:\text{Prop}, (A \rightarrow C) \rightarrow (B \rightarrow C) \rightarrow \text{or } A B \rightarrow C$.

Hypothesis

`or_elim_redl` :

$\forall (A B C:\text{Prop}) (f:A \rightarrow C) (g:B \rightarrow C) (a:A),$
 $f a = \text{or_elim } A B C f g (\text{or_introl } A B a).$

Hypothesis

`or_elim_redr` :

$\forall (A B C:\text{Prop}) (f:A \rightarrow C) (g:B \rightarrow C) (b:B),$
 $g b = \text{or_elim } A B C f g (\text{or_intror } A B b).$

Hypothesis

`or_dep_elim` :

$\forall (A B:\text{Prop}) (P:\text{or } A B \rightarrow \text{Prop}),$
 $(\forall a:A, P (\text{or_introl } A B a)) \rightarrow$
 $(\forall b:B, P (\text{or_intror } A B b)) \rightarrow \forall b:\text{or } A B, P b.$

Hypothesis `wem` : $\forall A:\text{Prop}, \text{or } (\sim \sim A) (\neg A).$

Variable `B` : **Prop**.

Variables `b1 b2` : `B`.

`p2b` and `b2p` form a retract if $\neg b1 = b2$

Let `p2b` (`A:NProp`) := `or_elim` (`~~El A`) (`~El A`) `B` (`fun _ => b1`) (`fun _ => b2`) (`wem (El A)`).

Let `b2p b` : `NProp` := `exist` (`fun P => ~~P -> P`) (`~~(b1 = b)`) (`fun h x => h (fun k => k x)`).

Lemma `wp2p1` : $\forall A:\text{NProp}, \text{El } A \rightarrow \text{El } (b2p (p2b A)).$

Lemma `wp2p2` : $b1 \neq b2 \rightarrow \forall A:\text{NProp}, \text{El } (b2p (p2b A)) \rightarrow \text{El } A.$

By Hurkens's paradox, we get a weak form of proof irrelevance.

Theorem `wproof_irrelevance_cc` : $\sim \sim (b1 = b2).$

End `Proof_irrelevance_WEM_CC`.

16.2.5 CIC |- excluded-middle -> proof-irrelevance

Since, dependent elimination is derivable in the Calculus of Inductive Constructions (CCI), we get proof-irrelevance from classical logic in the CCI.

Section Proof_irrelevance_CCI.

Hypothesis $em : \forall A:\text{Prop}, A \vee \neg A$.

Definition $or_elim_redl (A B C:\text{Prop}) (f:A \rightarrow C) (g:B \rightarrow C)$
 $(a:A) : f a = or_ind f g (or_introl B a) := eq_refl (f a)$.

Definition $or_elim_redr (A B C:\text{Prop}) (f:A \rightarrow C) (g:B \rightarrow C)$
 $(b:B) : g b = or_ind f g (or_intror A b) := eq_refl (g b)$.

Scheme $or_indd := \text{Induction for or Sort Prop}$.

Theorem $proof_irrelevance_cci : \forall (B:\text{Prop}) (b1 b2:B), b1 = b2$.

End Proof_irrelevance_CCI.

The same holds with weak excluded middle. The proof is a little more involved, however.

Section Weak_proof_irrelevance_CCI.

Hypothesis $wem : \forall A:\text{Prop}, \sim\sim A \vee \neg A$.

Theorem $wem_proof_irrelevance_cci : \forall (B:\text{Prop}) (b1 b2:B), \sim\sim b1 = b2$.

End Weak_proof_irrelevance_CCI.

Remark: in the Set-impredicative CCI, Hurkens' paradox still holds with *bool* in **Set** and since $\neg true = false$ for *true* and *false* in *bool* from **Set**, we get the inconsistency of $em : \forall A:\text{Prop}, \{A\} + \{\sim A\}$ in the Set-impredicative CCI.

16.3 Weak classical axioms

We show the following increasing in the strength of axioms:

- weak excluded-middle
- right distributivity of implication over disjunction and Gödel-Dummett axiom
- independence of general premises and drinker's paradox
- excluded-middle

16.3.1 Weak excluded-middle

The weak classical logic based on $\sim\sim A \vee \neg A$ is referred to with name KC in {*ChagrovZakharyashev97* [*ChagrovZakharyashev97*] Alexander Chagrov and Michael Zakharyashev, "Modal Logic", Clarendon Press, 1997.

Definition $weak_excluded_middle :=$
 $\forall A:\text{Prop}, \sim\sim A \vee \neg A$.

The interest in the equivalent variant *weak_generalized_excluded_middle* is that it holds even in logic without a primitive *False* connective (like Gödel-Dummett axiom)

Definition $weak_generalized_excluded_middle :=$
 $\forall A B:\text{Prop}, ((A \rightarrow B) \rightarrow B) \vee (A \rightarrow B)$.

16.3.2 Gödel-Dummett axiom

$(A \rightarrow B) \vee (B \rightarrow A)$ is studied in [Dummett59] and is based on [Gödel33].

[Dummett59] Michael A. E. Dummett. “A Propositional Calculus with a Denumerable Matrix”, In the Journal of Symbolic Logic, Vol 24 No. 2(1959), pp 97-103.

[Gödel33] Kurt Gödel. “Zum intuitionistischen Aussagenkalkül”, Ergeb. Math. Koll. 4 (1933), pp. 34-38.

Definition `GodelDummett` := $\forall A B:\text{Prop}, (A \rightarrow B) \vee (B \rightarrow A)$.

Lemma `excluded_middle_Godel_Dummett` : `excluded_middle` \rightarrow `GodelDummett`.

$(A \rightarrow B) \vee (B \rightarrow A)$ is equivalent to $(C \rightarrow A \vee B) \rightarrow (C \rightarrow A) \vee (C \rightarrow B)$ (proof from [Dummett59])

Definition `RightDistributivityImplicationOverDisjunction` :=

$\forall A B C:\text{Prop}, (C \rightarrow A \vee B) \rightarrow (C \rightarrow A) \vee (C \rightarrow B)$.

Lemma `Godel_Dummett_iff_right_distr_implication_over_disjunction` :

`GodelDummett` \leftrightarrow `RightDistributivityImplicationOverDisjunction`.

$(A \rightarrow B) \vee (B \rightarrow A)$ is stronger than the weak excluded middle

Lemma `Godel_Dummett_weak_excluded_middle` :

`GodelDummett` \rightarrow `weak_excluded_middle`.

16.3.3 Independence of general premises and drinker's paradox

Independence of general premises is the unconstrained, non constructive, version of the Independence of Premises as considered in [Troelstra73].

It is a generalization to predicate logic of the right distributivity of implication over disjunction (hence of Gödel-Dummett axiom) whose own constructive form (obtained by a restricting the third formula to be negative) is called Kreisel-Putnam principle [KreiselPutnam57].

[KreiselPutnam57], Georg Kreisel and Hilary Putnam. “Eine Unableitsbarkeitsbeweismethode für den intuitionistischen Aussagenkalkül”. Archiv für Mathematische Logik und Grundlagenforschung, 3:74- 78, 1957.

[Troelstra73], Anne Troelstra, editor. Metamathematical Investigation of Intuitionistic Arithmetic and Analysis, volume 344 of Lecture Notes in Mathematics, Springer-Verlag, 1973.

Definition `IndependenceOfGeneralPremises` :=

$\forall (A:\text{Type}) (P:A \rightarrow \text{Prop}) (Q:\text{Prop}),$
 $\text{inhabited } A \rightarrow (Q \rightarrow \exists x, P x) \rightarrow \exists x, Q \rightarrow P x.$

Lemma

`independence_general_premises_right_distr_implication_over_disjunction` :
`IndependenceOfGeneralPremises` \rightarrow `RightDistributivityImplicationOverDisjunction`.

Lemma `independence_general_premises_Godel_Dummett` :

`IndependenceOfGeneralPremises` \rightarrow `GodelDummett`.

Independence of general premises is equivalent to the drinker's paradox

Definition `DrinkerParadox` :=

$\forall (A:\text{Type}) (P:A \rightarrow \text{Prop}),$
 $\text{inhabited } A \rightarrow \exists x, (\exists x, P x) \rightarrow P x.$

Lemma independence_general_premises_drinker :

IndependenceOfGeneralPremises \leftrightarrow DrinkerParadox.

Independence of general premises is weaker than (generalized) excluded middle

Remark: generalized excluded middle is preferred here to avoid relying on the “ex falso quodlibet” property (i.e. $False \rightarrow \forall A, A$)

Definition generalized_excluded_middle :=

$\forall A B:\text{Prop}, A \vee (A \rightarrow B).$

Lemma excluded_middle_independence_general_premises :

generalized_excluded_middle \rightarrow DrinkerParadox.

Chapter 17

Library `Coq.Logic.Classical_Pred_Type`

Classical Predicate Logic on Type

`Require Import Classical_Prop.`

`Section Generic.`

`Variable U : Type.`

de Morgan laws for quantifiers

`Lemma not_all_not_ex :`

`∀ P:U → Prop, ¬ (∀ n:U, ¬ P n) → ∃ n : U, P n.`

`Lemma not_all_ex_not :`

`∀ P:U → Prop, ¬ (∀ n:U, P n) → ∃ n : U, ¬ P n.`

`Lemma not_ex_all_not :`

`∀ P:U → Prop, ¬ (∃ n : U, P n) → ∀ n:U, ¬ P n.`

`Lemma not_ex_not_all :`

`∀ P:U → Prop, ¬ (∃ n : U, ¬ P n) → ∀ n:U, P n.`

`Lemma ex_not_not_all :`

`∀ P:U → Prop, (∃ n : U, ¬ P n) → ¬ (∀ n:U, P n).`

`Lemma all_not_not_ex :`

`∀ P:U → Prop, (∀ n:U, ¬ P n) → ¬ (∃ n : U, P n).`

`End Generic.`

Chapter 18

Library `Coq.Logic.Classical_Prop`

Classical Propositional Logic

`Require Import ClassicalFacts.`

`Hint Unfold not: core.`

`Axiom classic : $\forall P:\text{Prop}, P \vee \neg P$.`

`Lemma NNPP : $\forall p:\text{Prop}, \neg \neg p \rightarrow p$.`

Peirce's law states $\forall P Q:\text{Prop}, ((P \rightarrow Q) \rightarrow P) \rightarrow P$. Thanks to $\forall P, \text{False} \rightarrow P$, it is equivalent to the following form

`Lemma Peirce : $\forall P:\text{Prop}, ((P \rightarrow \text{False}) \rightarrow P) \rightarrow P$.`

`Lemma not_imply_elim : $\forall P Q:\text{Prop}, \neg (P \rightarrow Q) \rightarrow P$.`

`Lemma not_imply_elim2 : $\forall P Q:\text{Prop}, \neg (P \rightarrow Q) \rightarrow \neg Q$.`

`Lemma imply_to_or : $\forall P Q:\text{Prop}, (P \rightarrow Q) \rightarrow \neg P \vee Q$.`

`Lemma imply_to_and : $\forall P Q:\text{Prop}, \neg (P \rightarrow Q) \rightarrow P \wedge \neg Q$.`

`Lemma or_to_imply : $\forall P Q:\text{Prop}, \neg P \vee Q \rightarrow P \rightarrow Q$.`

`Lemma not_and_or : $\forall P Q:\text{Prop}, \neg (P \wedge Q) \rightarrow \neg P \vee \neg Q$.`

`Lemma or_not_and : $\forall P Q:\text{Prop}, \neg P \vee \neg Q \rightarrow \neg (P \wedge Q)$.`

`Lemma not_or_and : $\forall P Q:\text{Prop}, \neg (P \vee Q) \rightarrow \neg P \wedge \neg Q$.`

`Lemma and_not_or : $\forall P Q:\text{Prop}, \neg P \wedge \neg Q \rightarrow \neg (P \vee Q)$.`

`Lemma imply_and_or : $\forall P Q:\text{Prop}, (P \rightarrow Q) \rightarrow P \vee Q \rightarrow Q$.`

`Lemma imply_and_or2 : $\forall P Q R:\text{Prop}, (P \rightarrow Q) \rightarrow P \vee R \rightarrow Q \vee R$.`

`Lemma proof_irrelevance : $\forall (P:\text{Prop}) (p1 p2:P), p1 = p2$.`

`Ltac classical_right := match goal with
| _ :- |-?X1 \vee _ => (elim (classic X1);intro;[left;trivial|right])
end.`

`Ltac classical_left := match goal with
| _ :- \vdash _ \wedge ?X1 => (elim (classic X1);intro;[right;trivial|left])`


```

end.

Require Export EqdepFacts.

Module EQ_RECT_EQ.

Lemma eq_rect_eq :
   $\forall (U:\text{Type}) (p:U) (Q:U \rightarrow \text{Type}) (x:Q\ p) (h:p = p), x = \text{eq\_rect } p\ Q\ x\ p\ h.$ 
End EQ_RECT_EQ.

Module EQDEPTHEORY := EQDEPTHEORY(EQ_RECT_EQ).
Export EqdepTheory.

```

Chapter 19

Library

Coq.Logic.ClassicalUniqueChoice

This file provides classical logic and unique choice; this is weaker than providing iota operator and classical logic as the definite descriptions provided by the axiom of unique choice can be used only in a propositional context (especially, they cannot be used to build functions outside the scope of a theorem proof)

Classical logic and unique choice, as shown in [ChicliPottierSimpson02], implies the double-negation of excluded-middle in **Set**, hence it implies a strongly classical world. Especially it conflicts with the impredicativity of **Set**.

[ChicliPottierSimpson02] Laurent Chicli, Loïc Pottier, Carlos Simpson, Mathematical Quotients and Quotient Types in Coq, Proceedings of TYPES 2002, Lecture Notes in Computer Science 2646, Springer Verlag.

Require Export Classical.

Axiom

dependent_unique_choice :
 $\forall (A:\text{Type}) (B:A \rightarrow \text{Type}) (R:\forall x:A, B\ x \rightarrow \text{Prop}),$
 $(\forall x:A, \exists! y:B\ x, R\ x\ y) \rightarrow$
 $(\exists f:(\forall x:A, B\ x), \forall x:A, R\ x\ (f\ x)).$

Unique choice reifies functional relations into functions

Theorem unique_choice :

$\forall (A\ B:\text{Type}) (R:A \rightarrow B \rightarrow \text{Prop}),$
 $(\forall x:A, \exists! y:B, R\ x\ y) \rightarrow$
 $(\exists f:A \rightarrow B, \forall x:A, R\ x\ (f\ x)).$

The following proof comes from [ChicliPottierSimpson02] **Require Import** Setoid.

Theorem classic_set_in_prop_context :

$\forall C:\text{Prop}, ((\forall P:\text{Prop}, \{P\} + \{\neg P\}) \rightarrow C) \rightarrow C.$

Corollary not_not_classic_set :

$((\forall P:\text{Prop}, \{P\} + \{\neg P\}) \rightarrow \text{False}) \rightarrow \text{False}.$

Notation classic_set := not_not_classic_set (*only parsing*).

Chapter 20

Library **Coq.Logic.Classical**

Classical Logic

Require Export Classical_Prop.

Require Export Classical_Pred_Type.

Chapter 21

Library `Coq.Logic.ConstructiveEpsilon`

This provides with a proof of the constructive form of definite and indefinite descriptions for Σ^0_1 -formulas (hereafter called “small” formulas), which infers the sigma-existence (i.e., **Type**-existence) of a witness to a decidable predicate over a countable domain from the regular existence (i.e., **Prop**-existence).

Coq does not allow case analysis on sort **Prop** when the goal is in not in **Prop**. Therefore, one cannot eliminate $\exists n, P\ n$ in order to show $\{n : \text{nat} \mid P\ n\}$. However, one can perform a recursion on an inductive predicate in sort **Prop** so that the returning type of the recursion is in **Type**. This trick is described in Coq’Art book, Sect. 14.2.3 and 15.4. In particular, this trick is used in the proof of *Fix_F* in the module `Coq.Init.Wf`. There, recursion is done on an inductive predicate *Acc* and the resulting type is in **Type**.

To find a witness of *P* constructively, we program the well-known linear search algorithm that tries *P* on all natural numbers starting from 0 and going up. Such an algorithm needs a suitable termination certificate. We offer two ways for providing this termination certificate: a direct one, based on an ad-hoc predicate called *before_witness*, and another one based on the predicate *Acc*. For the first one we provide explicit and short proof terms.

Based on ideas from Benjamin Werner and Jean-François Monin

Contributed by Yevgeniy Makarov and Jean-François Monin

Section `ConstructiveIndefiniteGroundDescription_Direct`.

Variable *P* : `nat` \rightarrow **Prop**.

Hypothesis *P_dec* : $\forall n, \{P\ n\} + \{\sim(P\ n)\}$.

The termination argument is *before_witness n*, which says that any number before any witness (not necessarily the *x* of $\exists x : A, P\ x$) makes the search eventually stops.

Inductive *before_witness* (*n*:`nat`) : **Prop** :=

| **stop** : *P* *n* \rightarrow *before_witness* *n*
| **next** : *before_witness* (**S** *n*) \rightarrow *before_witness* *n*.

Fixpoint *O_witness* (*n* : `nat`) : *before_witness* *n* \rightarrow *before_witness* 0 :=

match *n* **return** (*before_witness* *n* \rightarrow *before_witness* 0) **with**
| 0 \Rightarrow **fun** *b* \Rightarrow *b*
| **S** *n* \Rightarrow **fun** *b* \Rightarrow *O_witness* *n* (*next* *n* *b*)

end.

Definition inv_before_witness :

```

  ∀ n, before_witness n → ~ (P n) → before_witness (S n) :=
  fun n b =>
    match b return ¬ P n → before_witness (S n) with
    | stop _ p => fun not_p => match (not_p p) with end
    | next _ b => fun _ => b
    end.

```

Fixpoint linear_search m (b : before_witness m) : {n : nat | P n} :=

```

  match P_dec m with
  | left yes => exist (fun n => P n) m yes
  | right no => linear_search (S m) (inv_before_witness m b no)
  end.

```

Definition constructive_indefinite_ground_description_nat :

```

  (∃ n, P n) → {n : nat | P n} :=
  fun e => linear_search O (let (n, p) := e in O_witness n (stop n p)).

```

End ConstructiveIndefiniteGroundDescription_Direct.

Require Import Arith.

Section ConstructiveIndefiniteGroundDescription_Acc.

Variable P : nat → Prop.

Hypothesis P_decidable : ∀ n : nat, {P n} + {¬ P n}.

The predicate *Acc* delineates elements that are accessible via a given relation *R*. An element is accessible if there are no infinite *R*-descending chains starting from it.

To use *Fix_F*, we define a relation *R* and prove that if $\exists n, P n$ then 0 is accessible with respect to *R*. Then, by induction on the definition of *Acc* *R* 0, we show $\{n : nat \mid P n\}$.

The relation *R* describes the connection between the two successive numbers we try. Namely, *y* is *R*-less than *x* if we try *y* after *x*, i.e., $y = S x$ and *P* *x* is false. Then the absence of an infinite *R*-descending chain from 0 is equivalent to the termination of our searching algorithm.

Let *R* (*x y* : nat) : Prop := $x = S y \wedge \neg P y$.

Lemma P_implies_acc : ∀ *x* : nat, *P* *x* → acc *x*.

Lemma P_eventually_implies_acc : ∀ (*x* : nat) (*n* : nat), *P* (*n* + *x*) → acc *x*.

Corollary P_eventually_implies_acc_ex : (∃ *n* : nat, *P* *n*) → acc 0.

In the following statement, we use the trick with recursion on *Acc*. This is also where decidability of *P* is used.

Theorem acc_implies_P_eventually : acc 0 → {*n* : nat | *P* *n*}.

Theorem constructive_indefinite_ground_description_nat_Acc :

```

  (∃ n : nat, P n) → {n : nat | P n}.

```

End ConstructiveIndefiniteGroundDescription_Acc.

Section ConstructiveGroundEpsilon_nat.

Variable $P : \text{nat} \rightarrow \text{Prop}$.

Hypothesis $P_decidable : \forall x : \text{nat}, \{P\ x\} + \{\neg P\ x\}$.

Definition $\text{constructive_ground_epsilon_nat} (E : \exists n : \text{nat}, P\ n) : \text{nat}$
:= $\text{proj1_sig} (\text{constructive_indefinite_ground_description_nat } P\ P_decidable\ E)$.

Definition $\text{constructive_ground_epsilon_spec_nat} (E : (\exists n, P\ n)) : P (\text{constructive_ground_epsilon_nat } E)$
:= $\text{proj2_sig} (\text{constructive_indefinite_ground_description_nat } P\ P_decidable\ E)$.

End ConstructiveGroundEpsilon_nat.

Section ConstructiveGroundEpsilon.

For the current purpose, we say that a set A is countable if there are functions $f : A \rightarrow \text{nat}$ and $g : \text{nat} \rightarrow A$ such that g is a left inverse of f .

Variable $A : \text{Type}$.

Variable $f : A \rightarrow \text{nat}$.

Variable $g : \text{nat} \rightarrow A$.

Hypothesis $\text{gof_eq_id} : \forall x : A, g\ (f\ x) = x$.

Variable $P : A \rightarrow \text{Prop}$.

Hypothesis $P_decidable : \forall x : A, \{P\ x\} + \{\neg P\ x\}$.

Definition $P' (x : \text{nat}) : \text{Prop} := P\ (g\ x)$.

Lemma $P'_decidable : \forall n : \text{nat}, \{P'\ n\} + \{\neg P'\ n\}$.

Lemma $\text{constructive_indefinite_ground_description} : (\exists x : A, P\ x) \rightarrow \{x : A \mid P\ x\}$.

Lemma $\text{constructive_definite_ground_description} : (\exists! x : A, P\ x) \rightarrow \{x : A \mid P\ x\}$.

Definition $\text{constructive_ground_epsilon} (E : \exists x : A, P\ x) : A$
:= $\text{proj1_sig} (\text{constructive_indefinite_ground_description } E)$.

Definition $\text{constructive_ground_epsilon_spec} (E : (\exists x, P\ x)) : P (\text{constructive_ground_epsilon } E)$
:= $\text{proj2_sig} (\text{constructive_indefinite_ground_description } E)$.

End ConstructiveGroundEpsilon.

Chapter 22

Library `Coq.Logic.Decidable`

Properties of decidable propositions

Definition `decidable` ($P:\text{Prop}$) := $P \vee \neg P$.

Theorem `dec_not_not` : $\forall P:\text{Prop}, \text{decidable } P \rightarrow (\neg P \rightarrow \text{False}) \rightarrow P$.

Theorem `dec_True` : `decidable True`.

Theorem `dec_False` : `decidable False`.

Theorem `dec_or` :

$\forall A B:\text{Prop}, \text{decidable } A \rightarrow \text{decidable } B \rightarrow \text{decidable } (A \vee B)$.

Theorem `dec_and` :

$\forall A B:\text{Prop}, \text{decidable } A \rightarrow \text{decidable } B \rightarrow \text{decidable } (A \wedge B)$.

Theorem `dec_not` : $\forall A:\text{Prop}, \text{decidable } A \rightarrow \text{decidable } (\neg A)$.

Theorem `dec_imp` :

$\forall A B:\text{Prop}, \text{decidable } A \rightarrow \text{decidable } B \rightarrow \text{decidable } (A \rightarrow B)$.

Theorem `dec_iff` :

$\forall A B:\text{Prop}, \text{decidable } A \rightarrow \text{decidable } B \rightarrow \text{decidable } (A \leftrightarrow B)$.

Theorem `not_not` : $\forall P:\text{Prop}, \text{decidable } P \rightarrow \neg \neg P \rightarrow P$.

Theorem `not_or` : $\forall A B:\text{Prop}, \neg (A \vee B) \rightarrow \neg A \wedge \neg B$.

Theorem `not_and` : $\forall A B:\text{Prop}, \text{decidable } A \rightarrow \neg (A \wedge B) \rightarrow \neg A \vee \neg B$.

Theorem `not_imp` : $\forall A B:\text{Prop}, \text{decidable } A \rightarrow \neg (A \rightarrow B) \rightarrow A \wedge \neg B$.

Theorem `imp_simp` : $\forall A B:\text{Prop}, \text{decidable } A \rightarrow (A \rightarrow B) \rightarrow \neg A \vee B$.

Theorem `not_iff` :

$\forall A B:\text{Prop}, \text{decidable } A \rightarrow \text{decidable } B \rightarrow$
 $\neg (A \leftrightarrow B) \rightarrow (A \wedge \neg B) \vee (\neg A \wedge B)$.

Results formulated with `iff`, used in `FSetDecide`. Negation are expanded since it is unclear whether setoid rewrite will always perform conversion.

We begin with lemmas that, when read from left to right, can be understood as ways to eliminate uses of *not*.

Theorem `not_true_iff` : $(\text{True} \rightarrow \text{False}) \leftrightarrow \text{False}$.

```

Theorem not_false_iff : (False → False) ↔ True.
Theorem not_not_iff : ∀ A:Prop, decidable A →
  (((A → False) → False) ↔ A).
Theorem contrapositive : ∀ A B:Prop, decidable A →
  (((A → False) → (B → False)) ↔ (B → A)).
Lemma or_not_l_iff_1 : ∀ A B: Prop, decidable A →
  ((A → False) ∨ B ↔ (A → B)).
Lemma or_not_l_iff_2 : ∀ A B: Prop, decidable B →
  ((A → False) ∨ B ↔ (A → B)).
Lemma or_not_r_iff_1 : ∀ A B: Prop, decidable A →
  (A ∨ (B → False) ↔ (B → A)).
Lemma or_not_r_iff_2 : ∀ A B: Prop, decidable B →
  (A ∨ (B → False) ↔ (B → A)).
Lemma imp_not_l : ∀ A B: Prop, decidable A →
  (((A → False) → B) ↔ (A ∨ B)).

```

Moving Negations Around: We have four lemmas that, when read from left to right, describe how to push negations toward the leaves of a proposition and, when read from right to left, describe how to pull negations toward the top of a proposition.

```

Theorem not_or_iff : ∀ A B:Prop,
  (A ∨ B → False) ↔ (A → False) ∧ (B → False).
Lemma not_and_iff : ∀ A B:Prop,
  (A ∧ B → False) ↔ (A → B → False).
Lemma not_imp_iff : ∀ A B:Prop, decidable A →
  (((A → B) → False) ↔ A ∧ (B → False)).
Lemma not_imp_rev_iff : ∀ A B : Prop, decidable A →
  (((A → B) → False) ↔ (B → False) ∧ A).

```

```

Theorem dec_functional_relation :
  ∀ (X Y : Type) (A:X→Y→Prop), (∀ y y' : Y, decidable (y=y')) →
  (∀ x, ∃! y, A x y) → ∀ x y, decidable (A x y).

```

With the following hint database, we can leverage `auto` to check decidability of propositions.

```

Hint Resolve dec_True dec_False dec_or dec_and dec_imp dec_not dec_iff
: decidable_prop.

```

`solve_decidable using lib` will solve goals about the decidability of a proposition, assisted by an auxiliary database of lemmas. The database is intended to contain lemmas stating the decidability of base propositions, (e.g., the decidability of equality on a particular inductive type).

```

Tactic Notation "solve_decidable" "using" ident(db) :=
  match goal with
  | ⊢ decidable _ =>
    solve [ auto 100 with decidable_prop db ]
  end.

```



```
Tactic Notation "solve_decidable" :=  
  solve_decidable using core.
```

Chapter 23

Library **Coq.Logic.Description**

This file provides a constructive form of definite description; it allows building functions from the proof of their existence in any context; this is weaker than Church's iota operator

```
Require Import ChoiceFacts.
```

```
Set Implicit Arguments.
```

```
Axiom constructive_definite_description :
```

```
   $\forall (A : \mathbf{Type}) (P : A \rightarrow \mathbf{Prop}),$   
     $(\exists! x, P x) \rightarrow \{ x : A \mid P x \}.$ 
```

Chapter 24

Library **Coq.Logic.Diaconescu**

Diaconescu showed that the Axiom of Choice entails Excluded-Middle in topoi [Diaconescu75](#). Lacas and Werner adapted the proof to show that the axiom of choice in equivalence classes entails Excluded-Middle in Type Theory [LacasWerner99](#).

Three variants of Diaconescu's result in type theory are shown below.

A. A proof that the relational form of the Axiom of Choice + Extensionality for Predicates entails Excluded-Middle (by Hugo Herbelin)

B. A proof that the relational form of the Axiom of Choice + Proof Irrelevance entails Excluded-Middle for Equality Statements (by Benjamin Werner)

C. A proof that extensional Hilbert epsilon's description operator entails excluded-middle (taken from Bell [Bell93](#))

See also [Carlström](#) for a discussion of the connection between the Extensional Axiom of Choice and Excluded-Middle

[Diaconescu75](#) Radu Diaconescu, Axiom of Choice and Complementation, in Proceedings of AMS, vol 51, pp 176-178, 1975.

[LacasWerner99](#) Samuel Lacas, Benjamin Werner, Which Choices imply the excluded middle?, preprint, 1999.

[Bell93](#) John L. Bell, Hilbert's epsilon operator and classical logic, Journal of Philosophical Logic, 22: 1-18, 1993

[Carlström04](#) Jesper Carlström, EM + Ext + AC_int <-> AC_ext, Mathematical Logic Quarterly, vol 50(3), pp 236-240, 2004.

24.1 Pred. Ext. + Rel. Axiom of Choice -> Excluded-Middle

Section `PredExt_RelChoice_imp_EM`.

The axiom of extensionality for predicates

Definition `PredicateExtensionality` :=

$\forall P Q:\text{bool} \rightarrow \text{Prop}, (\forall b:\text{bool}, P\ b \leftrightarrow Q\ b) \rightarrow P = Q.$

From predicate extensionality we get propositional extensionality hence proof-irrelevance

Require Import `ClassicalFacts`.

Variable `pred_extensionality` : `PredicateExtensionality`.

Lemma prop_ext : $\forall A B:\text{Prop}, (A \leftrightarrow B) \rightarrow A = B$.

Lemma proof_irrel : $\forall (A:\text{Prop}) (a1\ a2:A), a1 = a2$.

From proof-irrelevance and relational choice, we get guarded relational choice

Require Import ChoiceFacts.

Variable rel_choice : RelationalChoice.

Lemma guarded_rel_choice : GuardedRelationalChoice.

The form of choice we need: there is a functional relation which chooses an element in any non empty subset of bool

Require Import Bool.

Lemma AC_bool_subset_to_bool :

$\exists R : (\text{bool} \rightarrow \text{Prop}) \rightarrow \text{bool} \rightarrow \text{Prop},$
 $(\forall P:\text{bool} \rightarrow \text{Prop},$
 $(\exists b : \text{bool}, P\ b) \rightarrow$
 $\exists b : \text{bool}, P\ b \wedge R\ P\ b \wedge (\forall b':\text{bool}, R\ P\ b' \rightarrow b = b')).$

The proof of the excluded middle Remark: P could have been in Set or Type

Theorem pred_ext_and_rel_choice_imp_EM : $\forall P:\text{Prop}, P \vee \neg P$.

End PredExt_RelChoice_imp_EM.

24.2 Proof-Irrel. + Rel. Axiom of Choice \rightarrow Excl.-Middle for Equality

This is an adaptation of Diaconescu's theorem, exploiting the form of extensionality provided by proof-irrelevance

Section ProofIrrel_RelChoice_imp_EqEM.

Variable rel_choice : RelationalChoice.

Variable proof_irrelevance : $\forall P:\text{Prop}, \forall x\ y:P, x=y$.

Let $a1$ and $a2$ be two elements in some type A

Variable A :Type.

Variables a1 a2 : A.

We build the subset A' of A made of $a1$ and $a2$

Definition A' := @sigT A (fun x \Rightarrow $x=a1 \vee x=a2$).

Definition a1':A'.

Defined.

Definition a2':A'.

Defined.

By proof-irrelevance, projection is a retraction

Lemma projT1_injective : $a1=a2 \rightarrow a1'=a2'$.

But from the actual proofs of being in A' , we can assert in the proof-irrelevant world the existence of relevant boolean witnesses

Lemma `decide` : $\forall x:A', \exists y:\text{bool},$
 $(\text{projT1 } x = a1 \wedge y = \text{true}) \vee (\text{projT1 } x = a2 \wedge y = \text{false}).$

Thanks to the axiom of choice, the boolean witnesses move from the propositional world to the relevant world

Theorem `proof_irrel_rel_choice_imp_eq_dec` : $a1=a2 \vee \neg a1=a2.$

An alternative more concise proof can be done by directly using the guarded relational choice

Lemma `proof_irrel_rel_choice_imp_eq_dec'` : $a1=a2 \vee \neg a1=a2.$

End `ProofIrrel_RelChoice_imp_EqEM`.

24.3 Extensional Hilbert's epsilon description operator \rightarrow Excluded-Middle

Proof sketch from Bell [Bell93](#) (with thanks to P. Castéran)

Section `ExtensionalEpsilon_imp_EM`.

Variable `epsilon` : $\forall A : \text{Type}, \text{inhabited } A \rightarrow (A \rightarrow \text{Prop}) \rightarrow A.$

Hypothesis `epsilon_spec` :

$\forall (A:\text{Type}) (i:\text{inhabited } A) (P:A\rightarrow\text{Prop}),$
 $(\exists x, P x) \rightarrow P (\text{epsilon } A i P).$

Hypothesis `epsilon_extensionality` :

$\forall (A:\text{Type}) (i:\text{inhabited } A) (P Q:A\rightarrow\text{Prop}),$
 $(\forall a, P a \leftrightarrow Q a) \rightarrow \text{epsilon } A i P = \text{epsilon } A i Q.$

Theorem `extensional_epsilon_imp_EM` : $\forall P:\text{Prop}, P \vee \neg P.$

End `ExtensionalEpsilon_imp_EM`.

Chapter 25

Library Coq.Logic.Epsilon

This file provides indefinite description under the form of Hilbert's epsilon operator; it does not assume classical logic.

Require Import ChoiceFacts.

Set Implicit Arguments.

Hilbert's epsilon: operator and specification in one statement

Axiom *epsilon_statement* :

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}), \text{inhabited } A \rightarrow$
 $\{ x : A \mid (\exists x, P x) \rightarrow P x \}.$

Lemma *constructive_indefinite_description* :

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}),$
 $(\exists x, P x) \rightarrow \{ x : A \mid P x \}.$

Lemma *small_drinkers'_paradox* :

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}), \text{inhabited } A \rightarrow$
 $\exists x, (\exists x, P x) \rightarrow P x.$

Theorem *iota_statement* :

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}), \text{inhabited } A \rightarrow$
 $\{ x : A \mid (\exists! x : A, P x) \rightarrow P x \}.$

Lemma *constructive_definite_description* :

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}),$
 $(\exists! x, P x) \rightarrow \{ x : A \mid P x \}.$

Hilbert's epsilon operator and its specification

Definition *epsilon* ($A : \text{Type}$) ($i : \text{inhabited } A$) ($P : A \rightarrow \text{Prop}$) : A
:= *proj1_sig* (*epsilon_statement* $P i$).

Definition *epsilon_spec* ($A : \text{Type}$) ($i : \text{inhabited } A$) ($P : A \rightarrow \text{Prop}$) :
 $(\exists x, P x) \rightarrow P$ (*epsilon* $i P$)
:= *proj2_sig* (*epsilon_statement* $P i$).

Church's iota operator and its specification

Definition *iota* ($A : \text{Type}$) ($i : \text{inhabited } A$) ($P : A \rightarrow \text{Prop}$) : A

```

:= proj1_sig (iota_statement  $P$   $i$ ).
Definition iota_spec ( $A : \text{Type}$ ) ( $i : \text{inhabited } A$ ) ( $P : A \rightarrow \text{Prop}$ ) :
  ( $\exists ! x : A, P\ x$ )  $\rightarrow P$  (iota  $i$   $P$ )
:= proj2_sig (iota_statement  $P$   $i$ ).

```

Chapter 26

Library `Coq.Logic.Eqdep_dec`

We prove that there is only one proof of $x=x$, i.e *eq_refl* x . This holds if the equality upon the set of x is decidable. A corollary of this theorem is the equality of the right projections of two equal dependent pairs.

Author: Thomas Kleymann |<tms@dcs.ed.ac.uk>| in Lego adapted to Coq by B. Barras

Credit: Proofs up to *K_dec* follow an outline by Michael Hedberg

Table of contents:

1. Streicher's K and injectivity of dependent pair hold on decidable types

1.1. Definition of the functor that builds properties of dependent equalities from a proof of decidability of equality for a set in Type

1.2. Definition of the functor that builds properties of dependent equalities from a proof of decidability of equality for a set in Set

26.1 Streicher's K and injectivity of dependent pair hold on decidable types

Set Implicit Arguments.

Section EqdepDec.

Variable $A : \text{Type}$.

Let $\text{comp} (x\ y\ y':A) (eq1:x = y) (eq2:x = y') : y = y' :=$
 $\text{eq_ind } _ (\text{fun } a \Rightarrow a = y') eq2 _ eq1.$

Remark $\text{trans_sym_eq} : \forall (x\ y:A) (u:x = y), \text{comp } u\ u = \text{eq_refl } y.$

Variable $x : A.$

Variable $\text{eq_dec} : \forall y:A, x = y \vee x \neq y.$

Let $\text{nu} (y:A) (u:x = y) : x = y :=$
 $\text{match } \text{eq_dec } y \text{ with}$
 $\quad | \text{or_introl } eqxy \Rightarrow eqxy$
 $\quad | \text{or_intror } neqxy \Rightarrow \text{False_ind } _ (neqxy\ u)$
end.

Let $nu_constant : \forall (y:A) (u\ v:x = y), nu\ u = nu\ v$.

Qed.

Let $nu_inv (y:A) (v:x = y) : x = y := comp\ (nu\ (eq_refl\ x))\ v$.

Remark $nu_left_inv_on : \forall (y:A) (u:x = y), nu_inv\ (nu\ u) = u$.

Theorem $eq_proofs_unicity_on : \forall (y:A) (p1\ p2:x = y), p1 = p2$.

Theorem $K_dec_on :$

$\forall P:x = x \rightarrow Prop, P\ (eq_refl\ x) \rightarrow \forall p:x = x, P\ p$.

The corollary

Let $proj (P:A \rightarrow Prop) (exP:ex\ P) (def:P\ x) : P\ x :=$

```
match exP with
| ex_intro _ x' prf =>
  match eq_dec x' with
  | or_introl eqprf => eq_ind x' P prf x (eq_sym eqprf)
  | _ => def
end
end.
```

Theorem $inj_right_pair_on :$

$\forall (P:A \rightarrow Prop) (y\ y':P\ x),$
 $ex_intro\ P\ x\ y = ex_intro\ P\ x\ y' \rightarrow y = y'.$

End EqdepDec.

Now we prove the versions that require decidable equality for the entire type rather than just on the given element. The rest of the file uses this total decidable equality. We could do everything using decidable equality at a point (because the induction rule for eq is really an induction rule for $\{ y : A \mid x = y \}$), but we don't currently, because changing everything would break backward compatibility and no-one has yet taken the time to define the pointed versions, and then re-define the non-pointed versions in terms of those.

Theorem $eq_proofs_unicity\ A\ (eq_dec : \forall x\ y : A, x = y \vee x \neq y) (x : A)$
 $: \forall (y:A) (p1\ p2:x = y), p1 = p2$.

Theorem $K_dec\ A\ (eq_dec : \forall x\ y : A, x = y \vee x \neq y) (x : A)$
 $: \forall P:x = x \rightarrow Prop, P\ (eq_refl\ x) \rightarrow \forall p:x = x, P\ p$.

Theorem $inj_right_pair\ A\ (eq_dec : \forall x\ y : A, x = y \vee x \neq y) (x : A)$
 $: \forall (P:A \rightarrow Prop) (y\ y':P\ x),$
 $ex_intro\ P\ x\ y = ex_intro\ P\ x\ y' \rightarrow y = y'.$

Require Import EqdepFacts.

We deduce axiom K for (decidable) types $Theorem\ K_dec_type :$

$\forall A:Type,$
 $(\forall x\ y:A, \{x = y\} + \{x \neq y\}) \rightarrow$
 $\forall (x:A) (P:x = x \rightarrow Prop), P\ (eq_refl\ x) \rightarrow \forall p:x = x, P\ p$.

Theorem $K_dec_set :$

$\forall A:Set,$
 $(\forall x\ y:A, \{x = y\} + \{x \neq y\}) \rightarrow$

$\forall (x:A) (P:x = x \rightarrow \text{Prop}), P (\text{eq_refl } x) \rightarrow \forall p:x = x, P p.$

We deduce the *eq_rect_eq* axiom for (decidable) types **Theorem** *eq_rect_eq_dec* :

$\forall A:\text{Type},$
 $(\forall x y:A, \{x = y\} + \{x \neq y\}) \rightarrow$
 $\forall (p:A) (Q:A \rightarrow \text{Type}) (x:Q p) (h:p = p), x = \text{eq_rect } p Q x p h.$

We deduce the injectivity of dependent equality for decidable types **Theorem** *eq_dep_eq_dec* :

$\forall A:\text{Type},$
 $(\forall x y:A, \{x = y\} + \{x \neq y\}) \rightarrow$
 $\forall (P:A \rightarrow \text{Type}) (p:A) (x y:P p), \text{eq_dep } A P p x p y \rightarrow x = y.$

Theorem *UIP_dec* :

$\forall (A:\text{Type}),$
 $(\forall x y:A, \{x = y\} + \{x \neq y\}) \rightarrow$
 $\forall (x y:A) (p1 p2:x = y), p1 = p2.$

Unset Implicit Arguments.

26.1.1 Definition of the functor that builds properties of dependent equalities on decidable sets in **Type**

The signature of decidable sets in **Type**

Module *Type* *DECIDABLETYPE*.

Axiom *eq_dec* : $\forall x y:U, \{x = y\} + \{x \neq y\}.$

End *DECIDABLETYPE*.

The module *DecidableEqDep* collects equality properties for decidable set in **Type**

Module *DECIDABLEEQDEP* (*M*:*DECIDABLETYPE*).

Import *M*.

Invariance by Substitution of Reflexive Equality Proofs

Lemma *eq_rect_eq* :
 $\forall (p:U) (Q:U \rightarrow \text{Type}) (x:Q p) (h:p = p), x = \text{eq_rect } p Q x p h.$

Injectivity of Dependent Equality

Theorem *eq_dep_eq* :
 $\forall (P:U \rightarrow \text{Type}) (p:U) (x y:P p), \text{eq_dep } U P p x p y \rightarrow x = y.$

Uniqueness of Identity Proofs (UIP)

Lemma *UIP* : $\forall (x y:U) (p1 p2:x = y), p1 = p2.$

Uniqueness of Reflexive Identity Proofs

Lemma *UIP_refl* : $\forall (x:U) (p:x = x), p = \text{eq_refl } x.$

Streicher's axiom K

Lemma *Streicher_K* :
 $\forall (x:U) (P:x = x \rightarrow \text{Prop}), P (\text{eq_refl } x) \rightarrow \forall p:x = x, P p.$

Injectivity of equality on dependent pairs in **Type**

Lemma `inj_pairT2` :
 $\forall (P:U \rightarrow \text{Type}) (p:U) (x y:P p),$
 $\text{existT } P p x = \text{existT } P p y \rightarrow x = y.$

Proof-irrelevance on subsets of decidable sets

Lemma `inj_pairP2` :
 $\forall (P:U \rightarrow \text{Prop}) (x:U) (p q:P x),$
 $\text{ex_intro } P x p = \text{ex_intro } P x q \rightarrow p = q.$

End `DECIDABLEEQDEP`.

26.1.2 Definition of the functor that builds properties of dependent equalities on decidable sets in `Set`

The signature of decidable sets in `Set`

Module `Type` `DECIDABLESET`.

Parameter `U:Set`.
Axiom `eq_dec` : $\forall x y:U, \{x = y\} + \{x \neq y\}.$

End `DECIDABLESET`.

The module *DecidableEqDepSet* collects equality properties for decidable set in `Set`

Module `DECIDABLEEQDEPSET` (`M:DECIDABLESET`).

Import `M`.
Module `N:=DECIDABLEEQDEP(M)`.

Invariance by Substitution of Reflexive Equality Proofs

Lemma `eq_rect_eq` :
 $\forall (p:U) (Q:U \rightarrow \text{Type}) (x:Q p) (h:p = p), x = \text{eq_rect } p Q x p h.$

Injectivity of Dependent Equality

Theorem `eq_dep_eq` :
 $\forall (P:U \rightarrow \text{Type}) (p:U) (x y:P p), \text{eq_dep } U P p x p y \rightarrow x = y.$

Uniqueness of Identity Proofs (UIP)

Lemma `UIP` : $\forall (x y:U) (p1 p2:x = y), p1 = p2.$

Uniqueness of Reflexive Identity Proofs

Lemma `UIP_refl` : $\forall (x:U) (p:x = x), p = \text{eq_refl } x.$

Streicher's axiom K

Lemma `Streicher_K` :
 $\forall (x:U) (P:x = x \rightarrow \text{Prop}), P (\text{eq_refl } x) \rightarrow \forall p:x = x, P p.$

Proof-irrelevance on subsets of decidable sets

Lemma `inj_pairP2` :
 $\forall (P:U \rightarrow \text{Prop}) (x:U) (p q:P x),$
 $\text{ex_intro } P x p = \text{ex_intro } P x q \rightarrow p = q.$

Injectivity of equality on dependent pairs in `Type`

Lemma inj_pair2 :

$\forall (P:U \rightarrow \text{Type}) (p:U) (x\ y:P\ p),$
 $\text{existT } P\ p\ x = \text{existT } P\ p\ y \rightarrow x = y.$

Injectivity of equality on dependent pairs with second component in **Type**

Notation inj_pairT2 := inj_pair2.

End DECIDABLEEQDEPSET.

From decidability to inj_pair2 Lemma inj_pair2_eq_dec : $\forall A:\text{Type}, (\forall x\ y:A, \{x=y\}+\{x\neq y\})$
 \rightarrow

$(\forall (P:A \rightarrow \text{Type}) (p:A) (x\ y:P\ p), \text{existT } P\ p\ x = \text{existT } P\ p\ y \rightarrow x = y).$

Examples of short direct proofs of unicity of reflexivity proofs on specific domains

Lemma UIP_refl_unit ($x : \text{tt} = \text{tt}$) : $x = \text{eq_refl tt}.$

Lemma UIP_refl_bool ($b:\text{bool}$) ($x : b = b$) : $x = \text{eq_refl}.$

Lemma UIP_refl_nat ($n:\text{nat}$) ($x : n = n$) : $x = \text{eq_refl}.$

Chapter 27

Library **Coq.Logic.EqdepFacts**

This file defines dependent equality and shows its equivalence with equality on dependent pairs (inhabiting sigma-types). It derives the consequence of axiomatizing the invariance by substitution of reflexive equality proofs and shows the equivalence between the 4 following statements

- Invariance by Substitution of Reflexive Equality Proofs.
- Injectivity of Dependent Equality
- Uniqueness of Identity Proofs
- Uniqueness of Reflexive Identity Proofs
- Streicher's Axiom K

These statements are independent of the calculus of constructions 2.

References:

1 T. Streicher, Semantical Investigations into Intensional Type Theory, Habilitationsschrift, LMU München, 1993. 2 M. Hofmann, T. Streicher, The groupoid interpretation of type theory, Proceedings of the meeting Twenty-five years of constructive type theory, Venice, Oxford University Press, 1998

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1. Definition of dependent equality and equivalence with equality of dependent pairs and with dependent pair of equalities
2. $\text{Eq_rect_eq} \leftrightarrow \text{Eq_dep_eq} \leftrightarrow \text{UIP} \leftrightarrow \text{UIP_refl} \leftrightarrow \text{K}$
3. Definition of the functor that builds properties of dependent equalities assuming axiom `eq_rect_eq`

27.1 Definition of dependent equality and equivalence with equality of dependent pairs

Import *EqNotations*.

Section *Dependent_Equality*.

Variable $U : \text{Type}$.
 Variable $P : U \rightarrow \text{Type}$.
 Dependent equality
 Inductive $\text{eq_dep} (p:U) (x:P p) : \forall q:U, P q \rightarrow \text{Prop} :=$
 $\text{eq_dep_intro} : \text{eq_dep } p x p x.$
 Hint Constructors eq_dep : core.
 Lemma $\text{eq_dep_refl} : \forall (p:U) (x:P p), \text{eq_dep } p x p x.$
 Lemma $\text{eq_dep_sym} :$
 $\forall (p q:U) (x:P p) (y:P q), \text{eq_dep } p x q y \rightarrow \text{eq_dep } q y p x.$
 Hint Immediate eq_dep_sym : core.
 Lemma $\text{eq_dep_trans} :$
 $\forall (p q r:U) (x:P p) (y:P q) (z:P r),$
 $\text{eq_dep } p x q y \rightarrow \text{eq_dep } q y r z \rightarrow \text{eq_dep } p x r z.$
 Scheme $\text{eq_indd} := \text{Induction for eq Sort Prop}.$
 Equivalent definition of dependent equality as a dependent pair of equalities
 Inductive $\text{eq_dep1} (p:U) (x:P p) (q:U) (y:P q) : \text{Prop} :=$
 $\text{eq_dep1_intro} : \forall h:q = p, x = \text{rew } h \text{ in } y \rightarrow \text{eq_dep1 } p x q y.$
 Lemma $\text{eq_dep1_dep} :$
 $\forall (p:U) (x:P p) (q:U) (y:P q), \text{eq_dep1 } p x q y \rightarrow \text{eq_dep } p x q y.$
 Lemma $\text{eq_dep_dep1} :$
 $\forall (p q:U) (x:P p) (y:P q), \text{eq_dep } p x q y \rightarrow \text{eq_dep1 } p x q y.$
 End Dependent_Equality.
 Dependent equality is equivalent to equality on dependent pairs
 Lemma $\text{eq_sigT_eq_dep} :$
 $\forall (U:\text{Type}) (P:U \rightarrow \text{Type}) (p q:U) (x:P p) (y:P q),$
 $\text{existT } P p x = \text{existT } P q y \rightarrow \text{eq_dep } p x q y.$
 Notation $\text{eq_sigS_eq_dep} := \text{eq_sigT_eq_dep (compat "8.2")}$.
 Lemma $\text{eq_dep_eq_sigT} :$
 $\forall (U:\text{Type}) (P:U \rightarrow \text{Type}) (p q:U) (x:P p) (y:P q),$
 $\text{eq_dep } p x q y \rightarrow \text{existT } P p x = \text{existT } P q y.$
 Lemma $\text{eq_sigT_iff_eq_dep} :$
 $\forall (U:\text{Type}) (P:U \rightarrow \text{Type}) (p q:U) (x:P p) (y:P q),$
 $\text{existT } P p x = \text{existT } P q y \leftrightarrow \text{eq_dep } p x q y.$
 Notation $\text{equiv_eqex_eqdep} := \text{eq_sigT_iff_eq_dep (only parsing)}$.
 Lemma $\text{eq_sig_eq_dep} :$
 $\forall (U:\text{Type}) (P:U \rightarrow \text{Prop}) (p q:U) (x:P p) (y:P q),$
 $\text{exist } P p x = \text{exist } P q y \rightarrow \text{eq_dep } p x q y.$
 Lemma $\text{eq_dep_eq_sig} :$
 $\forall (U:\text{Type}) (P:U \rightarrow \text{Prop}) (p q:U) (x:P p) (y:P q),$

$\text{eq_dep } p \ x \ q \ y \rightarrow \text{exist } P \ p \ x = \text{exist } P \ q \ y.$

Lemma eq_sig_iff_eq_dep :

$\forall (U:\text{Type}) (P:U \rightarrow \text{Prop}) (p \ q:U) (x:P \ p) (y:P \ q),$
 $\text{exist } P \ p \ x = \text{exist } P \ q \ y \leftrightarrow \text{eq_dep } p \ x \ q \ y.$

Dependent equality is equivalent to a dependent pair of equalities

Set Implicit Arguments.

Lemma eq_sigT_sig_eq : $\forall X \ P \ (x1 \ x2:X) \ H1 \ H2, \text{existT } P \ x1 \ H1 = \text{existT } P \ x2 \ H2 \leftrightarrow$
 $\{H:x1=x2 \mid \text{rew } H \text{ in } H1 = H2\}.$

Lemma eq_sigT_fst :

$\forall X \ P \ (x1 \ x2:X) \ H1 \ H2 \ (H:\text{existT } P \ x1 \ H1 = \text{existT } P \ x2 \ H2), x1 = x2.$

Lemma eq_sigT_snd :

$\forall X \ P \ (x1 \ x2:X) \ H1 \ H2 \ (H:\text{existT } P \ x1 \ H1 = \text{existT } P \ x2 \ H2), \text{rew } (\text{eq_sigT_fst } H) \text{ in } H1 =$
 $H2.$

Lemma eq_sig_fst :

$\forall X \ P \ (x1 \ x2:X) \ H1 \ H2 \ (H:\text{exist } P \ x1 \ H1 = \text{exist } P \ x2 \ H2), x1 = x2.$

Lemma eq_sig_snd :

$\forall X \ P \ (x1 \ x2:X) \ H1 \ H2 \ (H:\text{exist } P \ x1 \ H1 = \text{exist } P \ x2 \ H2), \text{rew } (\text{eq_sig_fst } H) \text{ in } H1 = H2.$

Unset Implicit Arguments.

Exported hints

Hint Resolve eq_dep_intro : *core*.

Hint Immediate eq_dep_sym : *core*.

27.2 Eq_rect_eq <-> Eq_dep_eq <-> UIP <-> UIP_refl <-> K

Section Equivalences .

Variable $U:\text{Type}$.

Invariance by Substitution of Reflexive Equality Proofs

Definition Eq_rect_eq_on $(p : U) (Q : U \rightarrow \text{Type}) (x : Q \ p) :=$

$\forall (h : p = p), x = \text{eq_rect } p \ Q \ x \ p \ h.$

Definition $\text{Eq_rect_eq} := \forall p \ Q \ x, \text{Eq_rect_eq_on } p \ Q \ x.$

Injectivity of Dependent Equality

Definition Eq_dep_eq_on $(P : U \rightarrow \text{Type}) (p : U) (x : P \ p) :=$

$\forall (y : P \ p), \text{eq_dep } p \ x \ p \ y \rightarrow x = y.$

Definition $\text{Eq_dep_eq} := \forall P \ p \ x, \text{Eq_dep_eq_on } P \ p \ x.$

Uniqueness of Identity Proofs (UIP)

Definition UIP_on_ $(x \ y : U) (p1 : x = y) :=$

$\forall (p2 : x = y), p1 = p2.$

Definition UIP_ $:= \forall x \ y \ p1, \text{UIP_on_ } x \ y \ p1.$

Uniqueness of Reflexive Identity Proofs

Definition `UIP_refl_on_` ($x : U$) :=

$\forall (p : x = x), p = \text{eq_refl } x.$

Definition `UIP_refl_` := $\forall x, \text{UIP_refl_on_ } x.$

Streicher's axiom K

Definition `Streicher_K_on_` ($x : U$) ($P : x = x \rightarrow \text{Prop}$) :=

$P (\text{eq_refl } x) \rightarrow \forall p : x = x, P p.$

Definition `Streicher_K_` := $\forall x P, \text{Streicher_K_on_ } x P.$

Injectivity of Dependent Equality is a consequence of Invariance by Substitution of Reflexive Equality Proof

Lemma `eq_rect_eq_on__eq_dep1_eq_on` ($p : U$) ($P : U \rightarrow \text{Type}$) ($y : P p$) :

$\text{Eq_rect_eq_on } p P y \rightarrow \forall (x : P p), \text{eq_dep1 } p x p y \rightarrow x = y.$

Lemma `eq_rect_eq__eq_dep1_eq` :

$\text{Eq_rect_eq} \rightarrow \forall (P : U \rightarrow \text{Type}) (p : U) (x y : P p), \text{eq_dep1 } p x p y \rightarrow x = y.$

Lemma `eq_rect_eq_on__eq_dep_eq_on` ($p : U$) ($P : U \rightarrow \text{Type}$) ($x : P p$) :

$\text{Eq_rect_eq_on } p P x \rightarrow \text{Eq_dep_eq_on } P p x.$

Lemma `eq_rect_eq__eq_dep_eq` : $\text{Eq_rect_eq} \rightarrow \text{Eq_dep_eq}.$

Uniqueness of Identity Proofs (UIP) is a consequence of Injectivity of Dependent Equality

Lemma `eq_dep_eq_on__UIP_on` ($x y : U$) ($p1 : x = y$) :

$\text{Eq_dep_eq_on } (\text{fun } y \Rightarrow x = y) x \text{eq_refl} \rightarrow \text{UIP_on_ } x y p1.$

Lemma `eq_dep_eq__UIP` : $\text{Eq_dep_eq} \rightarrow \text{UIP_}.$

Uniqueness of Reflexive Identity Proofs is a direct instance of UIP

Lemma `UIP_on__UIP_refl_on` ($x : U$) :

$\text{UIP_on_ } x x \text{eq_refl} \rightarrow \text{UIP_refl_on_ } x.$

Lemma `UIP__UIP_refl` : $\text{UIP_} \rightarrow \text{UIP_refl_}.$

Streicher's axiom K is a direct consequence of Uniqueness of Reflexive Identity Proofs

Lemma `UIP_refl_on__Streicher_K_on` ($x : U$) ($P : x = x \rightarrow \text{Prop}$) :

$\text{UIP_refl_on_ } x \rightarrow \text{Streicher_K_on_ } x P.$

Lemma `UIP_refl__Streicher_K` : $\text{UIP_refl_} \rightarrow \text{Streicher_K_}.$

We finally recover from K the Invariance by Substitution of Reflexive Equality Proofs

Lemma `Streicher_K_on__eq_rect_eq_on` ($p : U$) ($P : U \rightarrow \text{Type}$) ($x : P p$) :

$\text{Streicher_K_on_ } p (\text{fun } h \Rightarrow x = \text{rew} \rightarrow [P] h \text{ in } x)$

$\rightarrow \text{Eq_rect_eq_on } p P x.$

Lemma `Streicher_K__eq_rect_eq` : $\text{Streicher_K_} \rightarrow \text{Eq_rect_eq}.$

Remark: It is reasonable to think that `eq_rect_eq` is strictly stronger than `eq_rec_eq` (which is `eq_rect_eq` restricted on `Set`):

Definition `Eq_rec_eq` := $\forall (P : U \rightarrow \text{Set}) (p : U) (x : P p) (h : p = p), x = \text{eq_rec } p P x p h.$

Typically, `eq_rect_eq` allows proving UIP and Streicher's K what does not seem possible with `eq_rec_eq`. In particular, the proof of `UIP` requires to use `eq_rect_eq` on `fun y → x=y` which is in `Type` but not in `Set`.

End Equivalences.

UIP_refl is downward closed (a short proof of the key lemma of Voevodsky's proof of inclusion of h-level n into h-level n+1; see hlevelntosn in <https://github.com/vladimirias/Foundations.git>).

Theorem UIP_shift_on $(X : \text{Type}) (x : X) :$

UIP_refl_on_ $X x \rightarrow \forall y : x = x, \text{UIP_refl_on_} (x = x) y.$

Theorem UIP_shift : $\forall U, \text{UIP_refl_} U \rightarrow \forall x:U, \text{UIP_refl_} (x = x).$

Section Corollaries.

Variable $U:\text{Type}.$

UIP implies the injectivity of equality on dependent pairs in Type

Definition Inj_dep_pair_on $(P : U \rightarrow \text{Type}) (p : U) (x : P p) :=$

$\forall (y : P p), \text{existT } P p x = \text{existT } P p y \rightarrow x = y.$

Definition Inj_dep_pair := $\forall P p x, \text{Inj_dep_pair_on } P p x.$

Lemma eq_dep_eq_on_inj_pair2_on $(P : U \rightarrow \text{Type}) (p : U) (x : P p) :$

$\text{Eq_dep_eq_on } U P p x \rightarrow \text{Inj_dep_pair_on } P p x.$

Lemma eq_dep_eq_inj_pair2 : $\text{Eq_dep_eq } U \rightarrow \text{Inj_dep_pair}.$

End Corollaries.

Notation Inj_dep_pairS := Inj_dep_pair.

Notation Inj_dep_pairT := Inj_dep_pair.

Notation eq_dep_eq_inj_pairT2 := eq_dep_eq_inj_pair2.

27.3 Definition of the functor that builds properties of dependent equalities assuming axiom eq_rect_eq

Module Type EQDEPELIMINATION.

Axiom eq_rect_eq :

$\forall (U:\text{Type}) (p:U) (Q:U \rightarrow \text{Type}) (x:Q p) (h:p = p),$
 $x = \text{eq_rect } p Q x p h.$

End EQDEPELIMINATION.

Module EQDEPTHEORY $(M:\text{EQDEPELIMINATION}).$

Section Axioms.

Variable $U:\text{Type}.$

Invariance by Substitution of Reflexive Equality Proofs

Lemma eq_rect_eq :

$\forall (p:U) (Q:U \rightarrow \text{Type}) (x:Q p) (h:p = p), x = \text{eq_rect } p Q x p h.$

Lemma eq_rec_eq :

$\forall (p:U) (Q:U \rightarrow \text{Set}) (x:Q p) (h:p = p), x = \text{eq_rec } p Q x p h.$

Injectivity of Dependent Equality

Lemma eq_dep_eq : $\forall (P:U \rightarrow \text{Type}) (p:U) (x y:P p), \text{eq_dep } p x p y \rightarrow x = y.$

Uniqueness of Identity Proofs (UIP) is a consequence of Injectivity of Dependent Equality

Lemma `UIP` : $\forall (x\ y:U) (p1\ p2:x = y), p1 = p2$.

Uniqueness of Reflexive Identity Proofs is a direct instance of UIP

Lemma `UIP_refl` : $\forall (x:U) (p:x = x), p = \text{eq_refl } x$.

Streicher's axiom K is a direct consequence of Uniqueness of Reflexive Identity Proofs

Lemma `Streicher_K` :

$\forall (x:U) (P:x = x \rightarrow \text{Prop}), P (\text{eq_refl } x) \rightarrow \forall p:x = x, P\ p$.

End `Axioms`.

UIP implies the injectivity of equality on dependent pairs in Type

Lemma `inj_pair2` :

$\forall (U:\text{Type}) (P:U \rightarrow \text{Type}) (p:U) (x\ y:P\ p),$
 $\text{existT } P\ p\ x = \text{existT } P\ p\ y \rightarrow x = y$.

Notation `inj_pairT2` := `inj_pair2`.

End `EQDEPTHEORY`.

Basic facts about eq_dep

Lemma `f_eq_dep` :

$\forall U (P:U \rightarrow \text{Type}) R\ p\ q\ x\ y (f:\forall p, P\ p \rightarrow R\ p),$
 $\text{eq_dep } p\ x\ q\ y \rightarrow \text{eq_dep } p\ (f\ p\ x)\ q\ (f\ q\ y)$.

Lemma `eq_dep_non_dep` :

$\forall U\ P\ p\ q\ x\ y, @\text{eq_dep } U\ (\text{fun } _ \Rightarrow P)\ p\ x\ q\ y \rightarrow x = y$.

Lemma `f_eq_dep_non_dep` :

$\forall U (P:U \rightarrow \text{Type}) R\ p\ q\ x\ y (f:\forall p, P\ p \rightarrow R),$
 $\text{eq_dep } p\ x\ q\ y \rightarrow f\ p\ x = f\ q\ y$.

Chapter 28

Library **Coq.Logic.Eqdep**

This file axiomatizes the invariance by substitution of reflexive equality proofs [Streicher93] and exports its consequences, such as the injectivity of the projection of the dependent pair.

[Streicher93] T. Streicher, Semantical Investigations into Intensional Type Theory, Habilitationsschrift, LMU München, 1993.

```
Require Export EqdepFacts.
```

```
Module EQ_RECT_EQ.
```

```
Axiom eq_rect_eq :
```

```
   $\forall (U:\text{Type}) (p:U) (Q:U \rightarrow \text{Type}) (x:Q\ p) (h:p = p), x = \text{eq\_rect } p\ Q\ x\ p\ h.$ 
```

```
End EQ_RECT_EQ.
```

```
Module EQDEPTHEORY := EQDEPTHEORY(EQ_RECT_EQ).
```

```
Export EqdepTheory.
```

Exported hints

```
Hint Resolve eq_dep_eq: eqdep v62.
```

```
Hint Resolve inj_pair2 inj_pairT2: eqdep.
```

Chapter 29

Library **Coq.Logic.WeakFan**

A constructive proof of a non-standard version of the weak Fan Theorem in the formulation of which infinite paths are treated as predicates. The representation of paths as relations avoid the need for classical logic and unique choice. The idea of the proof comes from the proof of the weak König's lemma from separation in second-order arithmetic [*Simpson99*].

[*Simpson99*] Stephen G. Simpson. Subsystems of second order arithmetic, Cambridge University Press, 1999

Require Import List.

Import ListNotations.

inductively_barred P l means that P eventually holds above l

Inductive inductively_barred P : list bool → Prop :=

| **now** l : P l → inductively_barred P l

| **propagate** l :

 inductively_barred P (true::l) →

 inductively_barred P (false::l) →

 inductively_barred P l.

approx X l says that l is a boolean representation of a prefix of X

Fixpoint approx X (l:list bool) :=

match l **with**

 | [] ⇒ True

 | b::l ⇒ approx X l ∧ (if b then X (length l) else ¬ X (length l))

end.

barred P means that for any infinite path represented as a predicate, the property P holds for some prefix of the path

Definition barred P :=

 ∀ (X:nat → Prop), ∃ l, approx X l ∧ P l.

The proof proceeds by building a set Y of finite paths approximating either the smallest unbarred infinite path in P, if there is one (taking true>false), or the path true::true::... if P happens to be inductively_barred

Fixpoint Y P (l:list bool) :=

```

match l with
| [] ⇒ True
| b::l ⇒
    Y P l ∧
    if b then inductively_barred P (false::l) else ¬ inductively_barred P (false::l)
end.

Lemma Y-unique : ∀ P l1 l2, length l1 = length l2 → Y P l1 → Y P l2 → l1 = l2.

X is the translation of Y as a predicate
Definition X P n := ∃ l, length l = n ∧ Y P (true::l).

Lemma Y-approx : ∀ P l, approx (X P) l → Y P l.

Theorem WeakFanTheorem : ∀ P, barred P → inductively_barred P [].

```

Chapter 30

Library **Coq.Logic.WKL**

A constructive proof of a version of Weak König's Lemma over a decidable predicate in the formulation of which infinite paths are treated as predicates. The representation of paths as relations avoid the need for classical logic and unique choice. The decidability condition is sufficient to ensure that some required instance of double negation for disjunction of finite paths holds.

The idea of the proof comes from the proof of the weak König's lemma from separation in second-order arithmetic.

Notice that we do not start from a tree but just from an arbitrary predicate. Original Weak König's Lemma is the instantiation of the lemma to a tree

Require Import WeakFan List.

Import ListNotations.

Require Import Omega.

is_path_from $P\ n\ l$ means that there exists a path of length n from l on which P does not hold

Inductive *is_path_from* (P :list bool \rightarrow Prop) : nat \rightarrow list bool \rightarrow Prop :=
| **here** l : $\neg P\ l \rightarrow$ *is_path_from* $P\ 0\ l$
| **next_left** $l\ n$: $\neg P\ l \rightarrow$ *is_path_from* $P\ n\ (\text{true}::l) \rightarrow$ *is_path_from* $P\ (S\ n)\ l$
| **next_right** $l\ n$: $\neg P\ l \rightarrow$ *is_path_from* $P\ n\ (\text{false}::l) \rightarrow$ *is_path_from* $P\ (S\ n)\ l$.

We give the characterization of *is_path_from* in terms of a more common arithmetical formula

Proposition *is_path_from_characterization* $P\ n\ l$:

is_path_from $P\ n\ l \leftrightarrow \exists\ l', \text{length } l' = n \wedge \forall\ n', n' \leq n \rightarrow \neg P\ (\text{rev } (\text{firstn } n'\ l') ++ l).$

infinite_from $P\ l$ means that we can find arbitrary long paths along which P does not hold above l

Definition *infinite_from* (P :list bool \rightarrow Prop) l := $\forall\ n, \text{is_path_from } P\ n\ l$.

has_infinite_path P means that there is an infinite path (represented as a predicate) along which P does not hold at all

Definition *has_infinite_path* (P :list bool \rightarrow Prop) :=

$\exists\ (X:\text{nat} \rightarrow \text{Prop}), \forall\ l, \text{approx } X\ l \rightarrow \neg P\ l$.

inductively_barred_at $P\ n\ l$ means that P eventually holds above l after at most n steps upwards

Inductive *inductively_barred_at* (P :list bool \rightarrow Prop) : nat \rightarrow list bool \rightarrow Prop :=

```

| now_at l n : P l → inductively_barred_at P n l
| propagate_at l n :
  inductively_barred_at P n (true::l) →
  inductively_barred_at P n (false::l) →
  inductively_barred_at P (S n) l.

```

The proof proceeds by building a set Y of finite paths approximating either the smallest unbarred infinite path in P , if there is one (taking $true > false$), or the path $true::true::\dots$ if P happens to be inductively_barred

```

Fixpoint Y P (l:list bool) :=
  match l with
  | [] ⇒ True
  | b::l ⇒
    Y P l ∧
    if b then ∃ n, inductively_barred_at P n (false::l) else infinite_from P (false::l)
  end.

```

Require Import Compare_dec Le Lt.

```

Lemma is_path_from_restrict : ∀ P n n' l, n ≤ n' →
  is_path_from P n' l → is_path_from P n l.

```

```

Lemma inductively_barred_at_monotone : ∀ P l n n', n' ≤ n →
  inductively_barred_at P n' l → inductively_barred_at P n l.

```

```

Definition demorgan_or (P:list bool → Prop) l l' := ¬ (P l ∧ P l') → ¬ P l ∨ ¬ P l'.

```

```

Definition demorgan_inductively_barred_at P :=
  ∀ n l, demorgan_or (inductively_barred_at P n) (true::l) (false::l).

```

```

Lemma inductively_barred_at_imp_is_path_from :
  ∀ P, demorgan_inductively_barred_at P → ∀ n l,
  ¬ inductively_barred_at P n l → is_path_from P n l.

```

```

Lemma is_path_from_imp_inductively_barred_at : ∀ P n l,
  is_path_from P n l → inductively_barred_at P n l → False.

```

```

Lemma find_left_path : ∀ P l n,
  is_path_from P (S n) l → inductively_barred_at P n (false::l) → is_path_from P n (true::l).

```

```

Lemma Y_unique : ∀ P, demorgan_inductively_barred_at P →
  ∀ l1 l2, length l1 = length l2 → Y P l1 → Y P l2 → l1 = l2.

```

X is the translation of Y as a predicate

```

Definition X P n := ∃ l, length l = n ∧ Y P (true::l).

```

```

Lemma Y_approx : ∀ P, demorgan_inductively_barred_at P →
  ∀ l, approx (X P) l → Y P l.

```

Main theorem

```

Theorem PreWeakKonigsLemma : ∀ P,
  demorgan_inductively_barred_at P → infinite_from P [] → has_infinite_path P.

```

```

Lemma inductively_barred_at_decidable :

```

$\forall P, (\forall l, P\ l \vee \neg P\ l) \rightarrow \forall n\ l, \text{inductively_barred_at } P\ n\ l \vee \neg \text{inductively_barred_at } P\ n\ l.$

Lemma `inductively_barred_at_is_path_from_decidable` :

$\forall P, (\forall l, P\ l \vee \neg P\ l) \rightarrow \text{demorgan_inductively_barred_at } P.$

Main corollary

Corollary `WeakKonigsLemma` : $\forall P, (\forall l, P\ l \vee \neg P\ l) \rightarrow$
`infinite_from` $P\ [] \rightarrow \text{has_infinite_path } P.$

Chapter 31

Library

Coq.Logic.FunctionalExtensionality

This module states the axiom of (dependent) functional extensionality and (dependent) eta-expansion. It introduces a tactic **extensionality** to apply the axiom of extensionality to an equality goal.

The converse of functional extensionality.

Lemma `equal_f` : $\forall \{A\ B : \text{Type}\} \{f\ g : A \rightarrow B\},$
 $f = g \rightarrow \forall x, f\ x = g\ x.$

Lemma `equal_f_dep` : $\forall \{A\ B\} \{f\ g : \forall (x : A), B\ x\},$
 $f = g \rightarrow \forall x, f\ x = g\ x.$

Statements of functional extensionality for simple and dependent functions.

Axiom `functional_extensionality_dep` : $\forall \{A\} \{B : A \rightarrow \text{Type}\},$
 $\forall (f\ g : \forall x : A, B\ x),$
 $(\forall x, f\ x = g\ x) \rightarrow f = g.$

Lemma `functional_extensionality` $\{A\ B\} (f\ g : A \rightarrow B) :$
 $(\forall x, f\ x = g\ x) \rightarrow f = g.$

Extensionality of \forall s follows from functional extensionality. **Lemma** `forall_extensionality` $\{A\} \{B$
 $C : A \rightarrow \text{Type}\} (H : \forall x : A, B\ x = C\ x)$
 $: (\forall x, B\ x) = (\forall x, C\ x).$

Lemma `forall_extensionalityP` $\{A\} \{B\ C : A \rightarrow \text{Prop}\} (H : \forall x : A, B\ x = C\ x)$
 $: (\forall x, B\ x) = (\forall x, C\ x).$

Lemma `forall_extensionalityS` $\{A\} \{B\ C : A \rightarrow \text{Set}\} (H : \forall x : A, B\ x = C\ x)$
 $: (\forall x, B\ x) = (\forall x, C\ x).$

Apply `functional_extensionality`, introducing variable `x`.

Tactic Notation "extensionality" *ident*(`x`) :=
`match goal with`
 $[\vdash ?X = ?Y] \Rightarrow$
 `(apply (@functional_extensionality _ _ X Y) ||`
 `apply (@functional_extensionality_dep _ _ X Y) ||`
 `apply forall_extensionalityP ||`

```

    apply forall_extensionalityS ||
    apply forall_extensionality) ; intro  $x$ 
end.

```

Eta expansion follows from extensionality.

Lemma `eta_expansion_dep` $\{A\} \{B : A \rightarrow \text{Type}\} (f : \forall x : A, B\ x) :$
 $f = \text{fun } x \Rightarrow f\ x.$

Lemma `eta_expansion` $\{A\ B\} (f : A \rightarrow B) : f = \text{fun } x \Rightarrow f\ x.$

Chapter 32

Library `Coq.Logic.ExtensionalityFacts`

Some facts and definitions about extensionality

We investigate the relations between the following extensionality principles

- Functional extensionality
- Equality of projections from diagonal
- Unicity of inverse bijections
- Bijectivity of bijective composition

Table of contents

1. Definitions
2. Functional extensionality \leftrightarrow Equality of projections from diagonal
3. Functional extensionality \leftrightarrow Unicity of inverse bijections
4. Functional extensionality \leftrightarrow Bijectivity of bijective composition

`Set Implicit Arguments.`

32.1 Definitions

Being an inverse

Definition `is_inverse` $A\ B\ f\ g := (\forall a:A, g\ (f\ a) = a) \wedge (\forall b:B, f\ (g\ b) = b)$.

The diagonal over A and the one-one correspondence with A

Record `Delta` $A := \{ \text{pi1}:A; \text{pi2}:A; \text{eq}:\text{pi1}=\text{pi2} \}$.

Definition `delta` $\{A\}\ (a:A) := \{ | \text{pi1} := a; \text{pi2} := a; \text{eq} := \text{eq_refl}\ a \}$.

Lemma `diagonal_projs_same_behavior` : $\forall A\ (x:\text{Delta}\ A), \text{pi1}\ x = \text{pi2}\ x$.

Lemma `diagonal_inverse1` : $\forall A, \text{is_inverse}\ (A:=A)\ \text{delta}\ \text{pi1}$.

Lemma `diagonal_inverse2` : $\forall A, \text{is_inverse}\ (A:=A)\ \text{delta}\ \text{pi2}$.

Functional extensionality

Equality of projections from diagonal

Unicity of bijection inverse

Bijectivity of bijective composition

Definition `action A B C (f:A→B) := (fun h:B→C => fun x => h (f x)).`

32.2 Functional extensionality <-> Equality of projections from diagonal

Theorem `FuncExt_iff_EqDeltaProjs : FunctionalExtensionality <=> EqDeltaProjs.`

32.3 Functional extensionality <-> Unicity of bijection inverse

Lemma `FuncExt_UniqInverse : FunctionalExtensionality -> UniqueInverse.`

Lemma `UniqInverse_EqDeltaProjs : UniqueInverse -> EqDeltaProjs.`

Theorem `FuncExt_iff_UniqInverse : FunctionalExtensionality <=> UniqueInverse.`

32.4 Functional extensionality <-> Bijectivity of bijective composition

Lemma `FuncExt_BijComp : FunctionalExtensionality -> BijectivityBijectiveComp.`

Lemma `BijComp_FuncExt : BijectivityBijectiveComp -> FunctionalExtensionality.`

Chapter 33

Library `Coq.Logic.Hurkens`

Exploiting Hurkens’s paradox [Hurkens95] for system U- so as to derive various contradictory contexts.

The file is divided into various sub-modules which all follow the same structure: a section introduces the contradictory hypotheses and a theorem named *paradox* concludes the module with a proof of *False*.

- The *Generic* module contains the actual Hurkens’s paradox for a postulated shallow encoding of system U- in Coq. This is an adaptation by Arnaud Spiwack of a previous, more restricted implementation by Herman Geuvers. It is used to derive every other special cases of the paradox in this file.
- The *NoRetractToImpredicativeUniverse* module contains a simple and effective formulation by Herman Geuvers [Geuvers01] of a result by Thierry Coquand [Coquand90]. It states that no impredicative sort can contain a type of which it is a retract. This result implies that Coq with classical logic stated in impredicative Set is inconsistent and that classical logic stated in Prop implies proof-irrelevance (see *ClassicalFacts.v*)
- The *NoRetractFromSmallPropositionToProp* module is a specialisation of the *NoRetractToImpredicativeUniverse* module to the case where the impredicative sort is **Prop**.
- The *NoRetractToModalProposition* module is a strengthening of the *NoRetractFromSmallPropositionToProp* module. It shows that given a monadic modality (aka closure operator) M , the type of modal propositions (i.e. such that $M A \rightarrow A$) cannot be a retract of a modal proposition. It is an example of use of the paradox where the universes of system U- are not mapped to universes of Coq.
- The *NoRetractToNegativeProp* module is the specialisation of the *NoRetractFromSmallPropositionToProp* module where the modality is double-negation. This result implies that the principle of weak excluded middle ($\forall A, \sim\sim A \vee \sim A$) implies a weak variant of proof irrelevance.
- The *NoRetractFromTypeToProp* module proves that **Prop** cannot be a retract of a larger type.
- The *TypeNeqSmallType* module proves that **Type** is different from any smaller type.

- The *PropNeqType* module proves that **Prop** is different from any larger **Type**. It is an instance of the previous result.

References:

- Coquand90* T. Coquand, “Metamathematical Investigations of a Calculus of Constructions”, Proceedings of Logic in Computer Science (LICS’90), 1990.
- Hurkens95* A. J. Hurkens, “A simplification of Girard’s paradox”, Proceedings of the 2nd international conference Typed Lambda-Calculi and Applications (TLCA’95), 1995.
- Geuvers01* H. Geuvers, “Inconsistency of Classical Logic in Type Theory”, 2001, revised 2007 (see ¹).

33.1 A modular proof of Hurkens’s paradox.

It relies on an axiomatisation of a shallow embedding of system U- (i.e. types of U- are interpreted by types of Coq). The universes are encoded in a style, due to Martin-Löf, where they are given by a set of names and a family *El*:*Name*→**Type** which interprets each name into a type. This allows the encoding of universe to be decoupled from Coq’s universes. Dependent products and abstractions are similarly postulated rather than encoded as Coq’s dependent products and abstractions.

Module *GENERIC*.

Section *Paradox*.

33.1.1 Axiomatisation of impredicative universes in a Martin-Löf style

System U- has two impredicative universes. In the proof of the paradox they are slightly asymmetric (in particular the reduction rules of the small universe are not needed). Therefore, the axioms are duplicated allowing for a weaker requirement than the actual system U-.

Large universe

Variable *U1* : **Type**.

Variable *El1* : *U1* → **Type**.

Closure by small product **Variable** *Forall1* : $\forall u:U1, (El1\ u \rightarrow U1) \rightarrow U1$.

Notation " \forall_1 'x : A , B" := (*Forall1* A (**fun** x \Rightarrow B)).

Notation "A \rightarrow_1 B" := (*Forall1* A (**fun** _ \Rightarrow B)).

Variable *lam1* : $\forall u\ B, (\forall x:El1\ u, El1\ (B\ x)) \rightarrow El1\ (\forall_1\ x:u, B\ x)$.

Notation " λ_1 'x , u" := (*lam1* _ _ (**fun** x \Rightarrow u)).

Variable *app1* : $\forall u\ B\ (f:El1\ (Forall1\ u\ B))\ (x:El1\ u), El1\ (B\ x)$.

Notation "f \cdot_1 'x" := (*app1* _ _ f x).

Variable *beta1* : $\forall u\ B\ (f:\forall x:El1\ u, El1\ (B\ x))\ x,$
 $(\lambda_1\ y, f\ y) \cdot_1\ x = f\ x$.

¹<http://www.cs.ru.nl/~herman/PUBS/newnote.ps.gz>

Closure by large products $U1$ only needs to quantify over itself. **Variable** $ForallU1$: $(U1 \rightarrow U1) \rightarrow U1$.

Notation " \forall_2 " A, F := $(ForallU1 \text{ (fun } A \Rightarrow F))$.

Variable $lamU1$: $\forall F, (\forall A:U1, El1 (F A)) \rightarrow El1 (\forall_2 A, F A)$.

Notation " λ_2 " x, u := $(lamU1 \text{ _ (fun } x \Rightarrow u))$.

Variable $appU1$: $\forall F (f:El1(\forall_2 A, F A)) (A:U1), El1 (F A)$.

Notation " $f \cdot_1$ " $[A]$:= $(appU1 \text{ _ } f A)$.

Variable $betaU1$: $\forall F (f:\forall A:U1, El1 (F A)) A, (\lambda_2 x, f x) \cdot_1 [A] = f A$.

Small universe

The small universe is an element of the large one. **Variable** $u0$: $U1$.

Notation $U0$:= $(El1 u0)$.

Variable $El0$: $U0 \rightarrow \text{Type}$.

Closure by small product $U0$ does not need reduction rules **Variable** $Forall0$: $\forall u:U0, (El0 u \rightarrow U0) \rightarrow U0$.

Notation " \forall_0 " $x : A, B$:= $(Forall0 A \text{ (fun } x \Rightarrow B))$.

Notation " $A \rightarrow_0 B$ " := $(Forall0 A \text{ (fun _ } \Rightarrow B))$.

Variable $lam0$: $\forall u B, (\forall x:El0 u, El0 (B x)) \rightarrow El0 (\forall_0 x:u, B x)$.

Notation " λ_0 " x, u := $(lam0 \text{ _ _ (fun } x \Rightarrow u))$.

Variable $app0$: $\forall u B (f:El0 (Forall0 u B)) (x:El0 u), El0 (B x)$.

Notation " $f \cdot_0$ " x := $(app0 \text{ _ _ } f x)$.

Closure by large products **Variable** $ForallU0$: $\forall u:U1, (El1 u \rightarrow U0) \rightarrow U0$.

Notation " \forall_0^1 " $A : U, F$:= $(ForallU0 U \text{ (fun } A \Rightarrow F))$.

Variable $lamU0$: $\forall U F, (\forall A:El1 U, El0 (F A)) \rightarrow El0 (\forall_0^1 A:U, F A)$.

Notation " λ_0^1 " x, u := $(lamU0 \text{ _ _ (fun } x \Rightarrow u))$.

Variable $appU0$: $\forall U F (f:El0(\forall_0^1 A:U, F A)) (A:El1 U), El0 (F A)$.

Notation " $f \cdot_0$ " $[A]$:= $(appU0 \text{ _ _ } f A)$.

33.1.2 Automating the rewrite rules of our encoding.

Local Ltac *simplify* :=

(repeat rewrite ?*beta1*, ?*betaU1*);
lazy **beta**.

Local Ltac *simplify_in h* :=

(repeat rewrite ?*beta1*, ?*betaU1* in *h*);
lazy **beta** in *h*.

33.1.3 Hurkens's paradox.

An inhabitant of $U0$ standing for *False*. **Variable** F : $U0$.

Preliminary definitions

Definition $V : U1 := \forall_2 A, ((A \rightarrow_1 u0) \rightarrow_1 A \rightarrow_1 u0) \rightarrow_1 A \rightarrow_1 u0$.

Definition $U : U1 := V \rightarrow_1 u0$.

Definition $sb (z:El1\ V) : El1\ V := \lambda_2 A, \lambda_1 r, \lambda_1 a, r \cdot_1 (z \cdot_1 [A] \cdot_1 r) \cdot_1 a$.

Definition $le (i:El1\ (U \rightarrow_1 u0)) (x:El1\ U) : U0 :=$

$x \cdot_1 (\lambda_2 A, \lambda_1 r, \lambda_1 a, i \cdot_1 (\lambda_1 v, (sb\ v) \cdot_1 [A] \cdot_1 r \cdot_1 a))$.

Definition $le' : El1\ ((U \rightarrow_1 u0) \rightarrow_1 U \rightarrow_1 u0) := \lambda_1 i, \lambda_1 x, le\ i\ x$.

Definition $induct (i:El1\ (U \rightarrow_1 u0)) : U0 :=$

$\forall_0^1 x:U, le\ i\ x \rightarrow_0 i \cdot_1 x$.

Definition $WF : El1\ U := \lambda_1 z, (induct\ (z \cdot_1 [U] \cdot_1 le'))$.

Definition $I (x:El1\ U) : U0 :=$

$(\forall_0^1 i:U \rightarrow_1 u0, le\ i\ x \rightarrow_0 i \cdot_1 (\lambda_1 v, (sb\ v) \cdot_1 [U] \cdot_1 le' \cdot_1 x)) \rightarrow_0 F$

Proof

Lemma $\Omega : El0\ (\forall_0^1 i:U \rightarrow_1 u0, induct\ i \rightarrow_0 i \cdot_1 WF)$.

Proof.

```

refine ( $\lambda_0^1 i, \lambda_0 y, -$ ).
refine ( $y \cdot_0 [-] \cdot_0 -$ ).
unfold le, WF, induct. simplify.
refine ( $\lambda_0^1 x, \lambda_0 h0, -$ ). simplify.
refine ( $y \cdot_0 [-] \cdot_0 -$ ).
unfold le. simplify.
unfold sb at 1. simplify.
unfold le' at 1. simplify.
exact h0.

```

Qed.

Lemma $lemma1 : El0\ (induct\ (\lambda_1 u, I\ u))$.

Proof.

```

unfold induct.
refine ( $\lambda_0^1 x, \lambda_0 p, -$ ). simplify.
refine ( $\lambda_0 q, -$ ).
assert ( $El0\ (I\ (\lambda_1 v, (sb\ v) \cdot_1 [U] \cdot_1 le' \cdot_1 x)))$ ) as h.
{ generalize ( $q \cdot_0 [\lambda_1 u, I\ u] \cdot_0 p$ ). simplify.
  intros q'.
  exact q'. }
refine ( $h \cdot_0 -$ ).
refine ( $\lambda_0^1 i, -$ ).
refine ( $\lambda_0 h', -$ ).
generalize ( $q \cdot_0 [\lambda_1 y, i \cdot_1 (\lambda_1 v, (sb\ v) \cdot_1 [U] \cdot_1 le' \cdot_1 y)]$ ). simplify.
intros q'.
refine ( $q' \cdot_0 -$ ). clear q'.
unfold le at 1 in h'. simplify_in h'.

```



```

  unfold sb at 1 in h'. simplify_in h'.
  unfold le' at 1 in h'. simplify_in h'.
  exact h'.
Qed.

Lemma lemma2 :  $Elo$  ( $(\forall_0^1 i : U \rightarrow_1 u0, \text{induct } i \rightarrow_0 i_1 WF) \rightarrow_0 F$ ).
Proof.
  refine ( $\lambda_0 x, -$ ).
  assert ( $Elo$  ( $I$   $WF$ )) as h.
  { generalize ( $x_0[\lambda_1 u, I u]_0 \text{lemma1}$ ). simplify.
    intros q.
    exact q. }
  refine ( $h_0-$ ). clear h.
  refine ( $\lambda_0^1 i, \lambda_0 h0, -$ ).
  generalize ( $x_0[\lambda_1 y, i_1(\lambda_1 v, (sb\ v)_1[U]_1 le'_1 y)]$ ). simplify.
  intros q.
  refine ( $q_0-$ ). clear q.
  unfold le in h0. simplify_in h0.
  unfold WF in h0. simplify_in h0.
  exact h0.
Qed.

Theorem paradox :  $Elo$   $F$ .
Proof.
  exact (lemma2_0Omega).
Qed.

End Paradox.

  The paradox tactic can be called as a shortcut to use the paradox.  $Ltac\ paradox\ h :=$ 
  unshelve ( $\text{refine } ((\text{fun } h \Rightarrow -) (\text{paradox } \text{-----})))$ .

End GENERIC.

```

33.2 Impredicative universes are not retracts.

There can be no retract to an impredicative Coq universe from a smaller type. In this version of the proof, the impredicativity of the universe is postulated with a pair of functions from the universe to its type and back which commute with dependent product in an appropriate way.

Module NORETRACTTOIMPREDICATIVEUNIVERSE.

Section Paradox.

Let $U2 := \text{Type}$.

Let $U1 : U2 := \text{Type}$.

Variable $U0 : U1$.

$U1$ is impredicative

Variable $u2u1 : U2 \rightarrow U1$.

Hypothesis $u22u1_unit : \forall (c:U2), c \rightarrow u22u1\ c$.

$u22u1_counit$ and $u22u1_coherent$ only apply to dependent product so that the equations happen in the smaller $U1$ rather than $U2$. Indeed, it is not generally the case that one can project from a large universe to an impredicative universe and then get back the original type again. It would be too strong a hypothesis to require (in particular, it is not true of **Prop**). The formulation is reminiscent of the monadic characteristic of the projection from a large type to **Prop**. **Hypothesis** $u22u1_counit : \forall (F:U1 \rightarrow U1), u22u1\ (\forall A, F\ A) \rightarrow (\forall A, F\ A)$.

Hypothesis $u22u1_coherent : \forall (F:U1 \rightarrow U1) (f:\forall x:U1, F\ x) (x:U1),$
 $u22u1_counit\ _\text{ (}u22u1_unit\ _\text{ f)}\ x = f\ x$.

$U0$ is a retract of $U1$

Variable $u02u1 : U0 \rightarrow U1$.

Variable $u12u0 : U1 \rightarrow U0$.

Hypothesis $u12u0_unit : \forall (b:U1), b \rightarrow u02u1\ (u12u0\ b)$.

Hypothesis $u12u0_counit : \forall (b:U1), u02u1\ (u12u0\ b) \rightarrow b$.

33.2.1 Paradox

Theorem $paradox : \forall F:U1, F$.

Proof.

intros F .

Generic.paradox h .

Large universe + exact $U1$.

+ exact (fun $X \Rightarrow X$).

+ *cbn*. exact (fun $u\ F \Rightarrow \forall x:u, F\ x$).

+ *cbn*. exact (fun $_ \ x \Rightarrow x$).

+ *cbn*. exact (fun $_ \ x \Rightarrow x$).

+ *cbn*. exact (fun $F \Rightarrow u22u1\ (\forall x, F\ x)$).

+ *cbn*. exact (fun $_ \ x \Rightarrow u22u1_unit\ _ \ x$).

+ *cbn*. exact (fun $_ \ x \Rightarrow u22u1_counit\ _ \ x$).

Small universe + exact $U0$.

The interpretation of the small universe is the image of $U0$ in $U1$. + *cbn*. exact (fun $X \Rightarrow u02u1\ X$).

+ *cbn*. exact (fun $u\ F \Rightarrow u12u0\ (\forall x:(u02u1\ u), u02u1\ (F\ x))$).

+ *cbn*. exact (fun $u\ F \Rightarrow u12u0\ (\forall x:u, u02u1\ (F\ x))$).

+ *cbn*. exact (u12u0 F).

+ *cbn* in h .

exact (u12u0_counit $_ \ h$).

+ *cbn*. *easy*.

+ *cbn*. intros **. now rewrite $u22u1_coherent$.

+ *cbn*. intros $\times x$. exact (u12u0_unit $_ \ x$).

+ *cbn*. intros $\times x$. exact (u12u0_counit $_ \ x$).

+ *cbn*. intros $\times x$. exact (u12u0_unit $_ \ x$).

+ *cbn*. intros $\times x$. exact (u12u0_counit $_ \ x$).

Qed.

End Paradox.

End NORETRACTTOIMPREDICATIVEUNIVERSE.

33.3 Modal fragments of **Prop** are not retracts

In presence of a monadic modality on **Prop**, we can define a subset of **Prop** of modal propositions which is also a complete Heyting algebra. These cannot be a retract of a modal proposition. This is a case where the universe in system U- are not encoded as Coq universes.

Module NORETRACTTOMODALPROPOSITION.

33.3.1 Monadic modality

Section Paradox.

Variable $M : \text{Prop} \rightarrow \text{Prop}$.

Hypothesis $\text{unit} : \forall A:\text{Prop}, A \rightarrow M A$.

Hypothesis $\text{join} : \forall A:\text{Prop}, M (M A) \rightarrow M A$.

Hypothesis $\text{incr} : \forall A B:\text{Prop}, (A \rightarrow B) \rightarrow M A \rightarrow M B$.

Lemma $\text{strength} : \forall A (P:A \rightarrow \text{Prop}), M(\forall x:A, P x) \rightarrow \forall x:A, M(P x)$.

Proof.

eauto.

Qed.

33.3.2 The universe of modal propositions

Definition $\text{MProp} := \{ P:\text{Prop} \mid M P \rightarrow P \}$.

Definition $\text{El} : \text{MProp} \rightarrow \text{Prop} := @\text{proj1_sig} _ _$.

Lemma $\text{modal} : \forall P:\text{MProp}, M(\text{El } P) \rightarrow \text{El } P$.

Proof.

intros $[P m]$. cbn.

exact m .

Qed.

Definition $\text{Forall} \{A:\text{Type}\} (P:A \rightarrow \text{MProp}) : \text{MProp}$.

Proof.

unshelve (refine (exist _ _)).

+ exact $(\forall x:A, \text{El } (P x))$.

+ intros $h x$.

 eapply strength in h .

 eauto using modal .

Defined.

33.3.3 Retract of the modal fragment of **Prop** in a small type

The retract is axiomatized using logical equivalence as the equality on propositions.

```

Variable bool : MProp.
Variable p2b : MProp → El bool.
Variable b2p : El bool → MProp.
Hypothesis p2p1 : ∀ A:MProp, El (b2p (p2b A)) → El A.
Hypothesis p2p2 : ∀ A:MProp, El A → El (b2p (p2b A)).

```

33.3.4 Paradox

Theorem paradox : ∀ B:MProp, El B.

Proof.

```

intros B.
Generic.paradox h.
  Large universe    + exact MProp.
+ exact El.
+ exact (fun _ => Forall).
+ cbn. exact (fun _ _ f => f).
+ cbn. exact (fun _ _ f => f).
+ exact Forall.
+ cbn. exact (fun _ f => f).
+ cbn. exact (fun _ f => f).
  Small universe   + exact bool.
+ exact (fun b => El (b2p b)).
+ cbn. exact (fun _ F => p2b (Forall (fun x => b2p (F x)))).
+ exact (fun _ F => p2b (Forall (fun x => b2p (F x)))).
+ apply p2b.
  exact B.
+ cbn in h. auto.
+ cbn. easy.
+ cbn. easy.
+ cbn. auto.
+ cbn. intros × f.
  apply p2p1 in f. cbn in f.
  exact f.
+ cbn. auto.
+ cbn. intros × f.
  apply p2p1 in f. cbn in f.
  exact f.

```

Qed.

End Paradox.

End NORETRACTTOMODALPROPOSITION.

33.4 The negative fragment of Prop is not a retract

The existence in the pure Calculus of Constructions of a retract from the negative fragment of **Prop** into a negative proposition is inconsistent. This is an instance of the previous result.

Module NORETRACTTONEGATIVEPROP.

33.4.1 The universe of negative propositions.

Definition NProp := { P:Prop | $\sim\sim P \rightarrow P$ }.

Definition El : NProp → Prop := @proj1_sig _ _.

Section Paradox.

33.4.2 Retract of the negative fragment of Prop in a small type

The retract is axiomatized using logical equivalence as the equality on propositions.

Variable bool : NProp.

Variable p2b : NProp → El bool.

Variable b2p : El bool → NProp.

Hypothesis p2p1 : $\forall A:NProp, El (b2p (p2b A)) \rightarrow El A$.

Hypothesis p2p2 : $\forall A:NProp, El A \rightarrow El (b2p (p2b A))$.

33.4.3 Paradox

Theorem paradox : $\forall B:NProp, El B$.

Proof.

intros B.

unshelve (refine ((fun h ⇒ _) (NoRetractToModalProposition.paradox _ _ _ _ _))).

+ exact (fun P ⇒ $\sim\sim P$).

+ exact bool.

+ exact p2b.

+ exact b2p.

+ exact B.

+ exact h.

+ cbn. auto.

+ cbn. auto.

+ cbn. auto.

+ auto.

+ auto.

Qed.

End Paradox.

End NORETRACTTONEGATIVEPROP.

33.5 Prop is not a retract

The existence in the pure Calculus of Constructions of a retract from Prop into a small type of Prop is inconsistent. This is a special case of the previous result.

Module NORETRACTFROMSMALLPROPOSITIONTOPROP.

33.5.1 The universe of propositions.

Definition NProp := { P:Prop | P → P }.

Definition El : NProp → Prop := @proj1_sig _ _.

Section MParadox.

33.5.2 Retract of Prop in a small type, using the identity modality.

Variable bool : NProp.

Variable p2b : NProp → El bool.

Variable b2p : El bool → NProp.

Hypothesis p2p1 : ∀ A:NProp, El (b2p (p2b A)) → El A.

Hypothesis p2p2 : ∀ A:NProp, El A → El (b2p (p2b A)).

33.5.3 Paradox

Theorem mparadox : ∀ B:NProp, El B.

Proof.

intros B.

unshelve (refine ((fun h ⇒ _) (NoRetractToModalProposition.paradox _ _ _ _ _))).

+ exact (fun P ⇒ P).

+ exact bool.

+ exact p2b.

+ exact b2p.

+ exact B.

+ exact h.

+ cbn. auto.

+ cbn. auto.

+ cbn. auto.

+ auto.

+ auto.

Qed.

End MParadox.

Section Paradox.

33.5.4 Retract of Prop in a small type

The retract is axiomatized using logical equivalence as the equality on propositions. Variable bool : Prop.

Variable p2b : Prop → bool.

Variable b2p : bool → Prop.

Hypothesis p2p1 : ∀ A:Prop, b2p (p2b A) → A.

Hypothesis p2p2 : ∀ A:Prop, A → b2p (p2b A).

33.5.5 Paradox

Theorem `paradox` : $\forall B:\text{Prop}, B$.

Proof.

```

intros B.
unshelve (refine (mparadox (exist _ bool (fun x => x)) - - -
  (exist _ B (fun x => x)))).
+ intros p. red. red. exact (p2b (El p)).
+ cbn. intros b. red.  $\exists$  (b2p b). exact (fun x => x).
+ cbn. intros [A H]. cbn. apply p2p1.
+ cbn. intros [A H]. cbn. apply p2p2.

```

Qed.

End `Paradox`.

End `NORETRACTFROMSMALLPROPOSITIONTOPROP`.

33.6 Large universes are no retracts of `Prop`.

The existence in the Calculus of Constructions with universes of a retract from some `Type` universe into `Prop` is inconsistent.

Module `NORETRACTFROMTYPETOPROP`.

Definition `Type2` := `Type`.

Definition `Type1` := `Type` : `Type2`.

Section `Paradox`.

33.6.1 Assumption of a retract from `Type` into `Prop`

Variable `down` : `Type1` \rightarrow `Prop`.

Variable `up` : `Prop` \rightarrow `Type1`.

Hypothesis `up_down` : $\forall (A:\text{Type1}), \text{up } (\text{down } A) = A \text{ :> Type1}$.

33.6.2 Paradox

Theorem `paradox` : $\forall P:\text{Prop}, P$.

Proof.

```

intros P.
Generic.paradox h.
  Large universe.      + exact Type1.
+ exact (fun X => X).
+ cbn. exact (fun u F =>  $\forall x, F x$ ).
+ cbn. exact (fun _ _ x => x).
+ cbn. exact (fun _ _ x => x).
+ exact (fun F =>  $\forall A:\text{Prop}, F(\text{up } A)$ ).
+ cbn. exact (fun F f A => f (up A)).

```

```

+ cbn.
  intros F f A.
  specialize (f (down A)).
  rewrite up_down in f.
  exact f.
+ exact Prop.
+ cbn. exact (fun X ⇒ X).
+ cbn. exact (fun A P ⇒ ∀ x:A, P x).
+ cbn. exact (fun A P ⇒ ∀ x:A, P x).
+ cbn. exact P.
+ exact h.
+ cbn. easy.
+ cbn.
  intros F f A.
  destruct (up_down A). cbn.
  reflexivity.
+ cbn. exact (fun _ _ x ⇒ x).
+ cbn. exact (fun _ _ x ⇒ x).
+ cbn. exact (fun _ _ x ⇒ x).
+ cbn. exact (fun _ _ x ⇒ x).
Qed.
End Paradox.
End NORETRACTFROMTYPETOPROP.

```

33.7 $A \neq \text{Type}$

No Coq universe can be equal to one of its elements.

```

Module TYPENEQSMALLTYPE.
Unset Universe Polymorphism.
Section Paradox.

```

33.7.1 Universe U is equal to one of its elements.

```

Let U := Type.
Variable A:U.
Hypothesis h : U=A.

```

33.7.2 Universe U is a retract of A

The following context is actually sufficient for the paradox to hold. The hypothesis $h:U=A$ is only used to define *down*, *up* and *up_down*.

```

Let down (X:U) : A := @eq_rect _ _ (fun X ⇒ X) X _ h.
Let up (X:A) : U := @eq_rect_r _ _ (fun X ⇒ X) X _ h.

```


Lemma up_down : $\forall (X:U), up (down X) = X$.

Proof.

unfold up,down.

rewrite $\leftarrow h$.

reflexivity.

Qed.

Theorem paradox : False.

Proof.

Generic.paradox p.

Large universe + exact U.

+ exact (fun X \Rightarrow X).

+ cbn. exact (fun X F $\Rightarrow \forall x:X, F x$).

+ cbn. exact (fun _ _ x $\Rightarrow x$).

+ cbn. exact (fun _ _ x $\Rightarrow x$).

+ exact (fun F $\Rightarrow \forall x:A, F (up x)$).

+ cbn. exact (fun _ f \Rightarrow fun x:A $\Rightarrow f (up x)$).

+ cbn. intros $\times f X$.

specialize (f (down X)).

rewrite up_down in f.

exact f.

Small universe + exact A.

The interpretation of A as a universe is U. + cbn. exact up.

+ cbn. exact (fun _ F $\Rightarrow down (\forall x, up (F x))$).

+ cbn. exact (fun _ F $\Rightarrow down (\forall x, up (F x))$).

+ cbn. exact (down False).

+ rewrite up_down in p.

exact p.

+ cbn. easy.

+ cbn. intros ? f X.

destruct (up_down X). cbn.

reflexivity.

+ cbn. intros ? ? f.

rewrite up_down.

exact f.

+ cbn. intros ? ? f.

rewrite up_down in f.

exact f.

+ cbn. intros ? ? f.

rewrite up_down.

exact f.

+ cbn. intros ? ? f.

rewrite up_down in f.

exact f.

Qed.

End Paradox.

End TYPENEQSMALLTYPE.

33.8 Prop \neq Type.

Special case of *TypeNeqSmallType*.

Module PROPNEQTYPE.

Theorem paradox : Prop \neq Type.

Proof.

 intros *h*.

$unshelve$ (refine (TypeNeqSmallType.paradox - _)).

 + exact Prop.

 + *easy*.

Qed.

End PROPNEQTYPE.

Chapter 34

Library `Coq.Logic.IndefiniteDescription`

This file provides a constructive form of indefinite description that allows building choice functions; this is weaker than Hilbert's epsilon operator (which implies weakly classical properties) but stronger than the axiom of choice (which cannot be used outside the context of a theorem proof).

```
Require Import ChoiceFacts.
```

```
Set Implicit Arguments.
```

```
Axiom constructive_indefinite_description :
```

```
  ∀ (A : Type) (P : A → Prop),  
    (∃ x, P x) → { x : A | P x }.
```

```
Lemma constructive_definite_description :
```

```
  ∀ (A : Type) (P : A → Prop),  
    (∃! x, P x) → { x : A | P x }.
```

```
Lemma functional_choice :
```

```
  ∀ (A B : Type) (R : A → B → Prop),  
    (∀ x : A, ∃ y : B, R x y) →  
    (∃ f : A → B, ∀ x : A, R x (f x)).
```

Chapter 35

Library **Coq.Logic.JMeq**

John Major's Equality as proposed by Conor McBride

Reference:

McBride Elimination with a Motive, Proceedings of TYPES 2000, LNCS 2277, pp 197-216, 2002.

Set Implicit Arguments.

Inductive JMeq (A:Type) (x:A) : \forall B:Type, B \rightarrow Prop :=
JMeq_refl : JMeq x x.

Hint Resolve JMeq_refl.

Definition JMeq_hom {A : Type} (x y : A) := JMeq x y.

Lemma JMeq_sym : \forall (A B:Type) (x:A) (y:B), JMeq x y \rightarrow JMeq y x.

Hint Immediate JMeq_sym.

Lemma JMeq_trans :

\forall (A B C:Type) (x:A) (y:B) (z:C), JMeq x y \rightarrow JMeq y z \rightarrow JMeq x z.

Axiom JMeq_eq : \forall (A:Type) (x y:A), JMeq x y \rightarrow x = y.

Lemma JMeq_ind : \forall (A:Type) (x:A) (P:A \rightarrow Prop),
P x $\rightarrow \forall$ y, JMeq x y \rightarrow P y.

Lemma JMeq_rec : \forall (A:Type) (x:A) (P:A \rightarrow Set),
P x $\rightarrow \forall$ y, JMeq x y \rightarrow P y.

Lemma JMeq_rect : \forall (A:Type) (x:A) (P:A \rightarrow Type),
P x $\rightarrow \forall$ y, JMeq x y \rightarrow P y.

Lemma JMeq_ind_r : \forall (A:Type) (x:A) (P:A \rightarrow Prop),
P x $\rightarrow \forall$ y, JMeq y x \rightarrow P y.

Lemma JMeq_rec_r : \forall (A:Type) (x:A) (P:A \rightarrow Set),
P x $\rightarrow \forall$ y, JMeq y x \rightarrow P y.

Lemma JMeq_rect_r : \forall (A:Type) (x:A) (P:A \rightarrow Type),
P x $\rightarrow \forall$ y, JMeq y x \rightarrow P y.

Lemma JMeq_congr :

$\forall (A:\text{Type}) (x:A) (B:\text{Type}) (f:A \rightarrow B) (y:A), \text{JMeq } x \ y \rightarrow f \ x = f \ y.$
 JMeq is equivalent to $\text{eq_dep Type (fun } X \Rightarrow X)$

Require Import Eqdep.

Lemma JMeq_eq_dep_id :

$\forall (A \ B:\text{Type}) (x:A) (y:B), \text{JMeq } x \ y \rightarrow \text{eq_dep Type (fun } X \Rightarrow X) \ A \ x \ B \ y.$

Lemma eq_dep_id_JMeq :

$\forall (A \ B:\text{Type}) (x:A) (y:B), \text{eq_dep Type (fun } X \Rightarrow X) \ A \ x \ B \ y \rightarrow \text{JMeq } x \ y.$

$\text{eq_dep } U \ P \ p \ x \ q \ y$ is strictly finer than $\text{JMeq } (P \ p) \ x \ (P \ q) \ y$

Lemma eq_dep_JMeq :

$\forall U \ P \ p \ x \ q \ y, \text{eq_dep } U \ P \ p \ x \ q \ y \rightarrow \text{JMeq } x \ y.$

Lemma eq_dep_strictly_stronger_JMeq :

$\exists U \ P \ p \ q \ x \ y, \text{JMeq } x \ y \wedge \neg \text{eq_dep } U \ P \ p \ x \ q \ y.$

However, when the dependencies are equal, $\text{JMeq } (P \ p) \ x \ (P \ q) \ y$ is as strong as $\text{eq_dep } U \ P \ p \ x \ q \ y$ (this uses JMeq_eq)

Lemma JMeq_eq_dep :

$\forall U \ (P:U \rightarrow \text{Prop}) \ p \ q \ (x:P \ p) \ (y:P \ q),$
 $p = q \rightarrow \text{JMeq } x \ y \rightarrow \text{eq_dep } U \ P \ p \ x \ q \ y.$

Notation sym_JMeq := JMeq_sym (*only parsing*).

Notation trans_JMeq := JMeq_trans (*only parsing*).

Chapter 36

Library **Coq.Logic.ProofIrrelevanceFacts**

This defines the functor that build consequences of proof-irrelevance

```
Require Export EqdepFacts.
```

```
Module Type PROOFIRRELEVANCE.
```

```
  Axiom proof_irrelevance :  $\forall (P:\text{Prop}) (p1\ p2:P), p1 = p2$ .
```

```
End PROOFIRRELEVANCE.
```

```
Module PROOFIRRELEVANCETHEORY (M:PROOFIRRELEVANCE).
```

Proof-irrelevance implies uniqueness of reflexivity proofs

```
Module EQ_RECT_EQ.
```

```
  Lemma eq_rect_eq :
```

```
     $\forall (U:\text{Type}) (p:U) (Q:U \rightarrow \text{Type}) (x:Q\ p) (h:p = p),$   
     $x = \text{eq\_rect}\ p\ Q\ x\ p\ h$ .
```

```
End EQ_RECT_EQ.
```

Export the theory of injective dependent elimination

```
Module EQDEPTHEORY := EQDEPTHEORY(EQ_RECT_EQ).
```

```
Export EqdepTheory.
```

```
Scheme eq_indd := Induction for eq Sort Prop.
```

We derive the irrelevance of the membership property for subsets

```
Lemma subset_eq_compat :
```

```
   $\forall (U:\text{Type}) (P:U \rightarrow \text{Prop}) (x\ y:U) (p:P\ x) (q:P\ y),$   
   $x = y \rightarrow \text{exist}\ P\ x\ p = \text{exist}\ P\ y\ q$ .
```

```
Lemma subsetT_eq_compat :
```

```
   $\forall (U:\text{Type}) (P:U \rightarrow \text{Prop}) (x\ y:U) (p:P\ x) (q:P\ y),$   
   $x = y \rightarrow \text{existT}\ P\ x\ p = \text{existT}\ P\ y\ q$ .
```

```
End PROOFIRRELEVANCETHEORY.
```

Chapter 37

Library **Coq.Logic.ProofIrrelevance**

This file axiomatizes proof-irrelevance and derives some consequences

```
Require Import ProofIrrelevanceFacts.
```

```
Axiom proof_irrelevance :  $\forall (P:\mathbf{Prop}) (p1\ p2:P), p1 = p2$ .
```

```
Module PI. Definition proof_irrelevance := proof_irrelevance. End PI.
```

```
Module PROOFIRRELEVANCETHEORY := PROOFIRRELEVANCETHEORY(PI).
```

```
Export ProofIrrelevanceTheory.
```

Chapter 38

Library **Coq.Logic.RelationalChoice**

This file axiomatizes the relational form of the axiom of choice

```
Axiom relational_choice :  
   $\forall (A\ B : \text{Type}) (R : A \rightarrow B \rightarrow \text{Prop}),$   
     $(\forall x : A, \exists y : B, R\ x\ y) \rightarrow$   
       $\exists R' : A \rightarrow B \rightarrow \text{Prop},$   
        subrelation  $R'\ R \wedge \forall x : A, \exists! y : B, R'\ x\ y.$ 
```


Chapter 39

Library `Coq.Logic.SetIsType`

39.1 The Set universe seen as a synonym for Type

After loading this file, Set becomes just another name for Type. This allows easily performing a Set-to-Type migration, or at least test whether a development relies or not on specific features of Set: simply insert some Require Export of this file at starting points of the development and try to recompile...

Notation "'Set'" := **Type** (*only parsing*).

Chapter 40

Library **Coq.Logic.FinFun**

40.1 Functions on finite domains

Main result : for functions $f:A \rightarrow B$ with finite A , f injective \leftrightarrow f bijective \leftrightarrow f surjective.

Require Import List Compare_dec EqNat Decidable ListDec. **Require** Fin.
Set Implicit Arguments.

General definitions

Definition Injective $\{A B\}$ $(f : A \rightarrow B) :=$
 $\forall x y, f x = f y \rightarrow x = y.$

Definition Surjective $\{A B\}$ $(f : A \rightarrow B) :=$
 $\forall y, \exists x, f x = y.$

Definition Bijective $\{A B\}$ $(f : A \rightarrow B) :=$
 $\exists g:B \rightarrow A, (\forall x, g (f x) = x) \wedge (\forall y, f (g y) = y).$

Finiteness is defined here via exhaustive list enumeration

Definition Full $\{A:\text{Type}\}$ $(l:\text{list } A) := \forall a:A, \text{In } a \text{ } l.$

Definition Finite $(A:\text{Type}) := \exists (l:\text{list } A), \text{Full } l.$

In many following proofs, it will be convenient to have list enumerations without duplicates. As soon as we have decidability of equality (in Prop), this is equivalent to the previous notion.

Definition Listing $\{A:\text{Type}\}$ $(l:\text{list } A) := \text{NoDup } l \wedge \text{Full } l.$

Definition Finite' $(A:\text{Type}) := \exists (l:\text{list } A), \text{Listing } l.$

Lemma Finite_alt $A (d:\text{decidable_eq } A) : \text{Finite } A \leftrightarrow \text{Finite' } A.$

Injections characterized in term of lists

Lemma Injective_map_NoDup $A B (f:A \rightarrow B) (l:\text{list } A) :$
 $\text{Injective } f \rightarrow \text{NoDup } l \rightarrow \text{NoDup } (\text{map } f \text{ } l).$

Lemma Injective_list_carac $A B (d:\text{decidable_eq } A)(f:A \rightarrow B) :$
 $\text{Injective } f \leftrightarrow (\forall l, \text{NoDup } l \rightarrow \text{NoDup } (\text{map } f \text{ } l)).$

Lemma Injective_carac $A B (l:\text{list } A) : \text{Listing } l \rightarrow$
 $\forall (f:A \rightarrow B), \text{Injective } f \leftrightarrow \text{NoDup } (\text{map } f \text{ } l).$

Surjection characterized in term of lists

Lemma `Surjective_list_carac` $A\ B\ (f:A \rightarrow B)$:
 $\text{Surjective } f \leftrightarrow (\forall lB, \exists lA, \text{incl } lB\ (\text{map } f\ lA)).$

Lemma `Surjective_carac` $A\ B : \text{Finite } B \rightarrow \text{decidable_eq } B \rightarrow$
 $\forall f:A \rightarrow B, \text{Surjective } f \leftrightarrow (\exists lA, \text{Listing } (\text{map } f\ lA)).$

Main result :

Lemma `Endo_Injective_Surjective` :
 $\forall A, \text{Finite } A \rightarrow \text{decidable_eq } A \rightarrow$
 $\forall f:A \rightarrow A, \text{Injective } f \leftrightarrow \text{Surjective } f.$

An injective and surjective function is bijective. We need here stronger hypothesis : decidability of equality in Type.

Definition `EqDec` $(A:\text{Type}) := \forall x\ y:A, \{x=y\} + \{x \neq y\}.$

First, we show that a surjective f has an inverse function g such that $f.g = \text{id}$.

Lemma `Finite_Empty_or_not` A :
 $\text{Finite } A \rightarrow (A \rightarrow \text{False}) \vee \exists a:A, \text{True}.$

Lemma `Surjective_inverse` :
 $\forall A\ B, \text{Finite } A \rightarrow \text{EqDec } B \rightarrow$
 $\forall f:A \rightarrow B, \text{Surjective } f \rightarrow$
 $\exists g:B \rightarrow A, \forall x, f\ (g\ x) = x.$

Same, with more knowledge on the inverse function: $g.f = f.g = \text{id}$

Lemma `Injective_Surjective_Bijective` :
 $\forall A\ B, \text{Finite } A \rightarrow \text{EqDec } B \rightarrow$
 $\forall f:A \rightarrow B, \text{Injective } f \rightarrow \text{Surjective } f \rightarrow \text{Bijective } f.$

An example of finite type : *Fin.t*

Lemma `Fin_Finite` $n : \text{Finite } (\text{Fin.t } n).$

Instead of working on a finite subset of nat, another solution is to use restricted $\text{nat} \rightarrow \text{nat}$ functions, and to consider them only below a certain bound n .

Definition `bFun` $n\ (f:\text{nat} \rightarrow \text{nat}) := \forall x, x < n \rightarrow f\ x < n.$

Definition `bInjective` $n\ (f:\text{nat} \rightarrow \text{nat}) :=$
 $\forall x\ y, x < n \rightarrow y < n \rightarrow f\ x = f\ y \rightarrow x = y.$

Definition `bSurjective` $n\ (f:\text{nat} \rightarrow \text{nat}) :=$
 $\forall y, y < n \rightarrow \exists x, x < n \wedge f\ x = y.$

We show that this is equivalent to the use of *Fin.t n*.

Module `FIN2RESTRICT`.

Notation `n2f` $:= \text{Fin.of_nat_lt}.$

Definition `f2n` $\{n\}\ (x:\text{Fin.t } n) := \text{proj1_sig } (\text{Fin.to_nat } x).$

Definition `f2n_ok` $n\ (x:\text{Fin.t } n) : \text{f2n } x < n := \text{proj2_sig } (\text{Fin.to_nat } x).$

Definition `n2f_f2n` $:= \forall n\ x, \text{n2f } (\text{f2n_ok } x) = x := @\text{Fin.of_nat_to_nat_inv}.$

Definition $\text{f2n_n2f } x \ n \ h : \text{f2n } (\text{n2f } h) = x := \text{f_equal } (\text{@proj1_sig } _ _) (\text{@Fin.to_nat_of_nat } x \ n \ h)$.

Definition $\text{n2f_ext} : \forall x \ n \ h \ h', \text{n2f } h = \text{n2f } h' := \text{@Fin.of_nat_ext}$.

Definition $\text{f2n_inj} : \forall n \ x \ y, \text{f2n } x = \text{f2n } y \rightarrow x = y := \text{@Fin.to_nat_inj}$.

Definition $\text{extend } n \ (f : \text{Fin.t } n \rightarrow \text{Fin.t } n) : (\text{nat} \rightarrow \text{nat}) :=$

```

fun x =>
  match le_lt_dec n x with
  | left _ => 0
  | right h => f2n (f (n2f h))
end.

```

Definition $\text{restrict } n \ (f : \text{nat} \rightarrow \text{nat}) (hf : \text{bFun } n \ f) : (\text{Fin.t } n \rightarrow \text{Fin.t } n) :=$

```

fun x => let (x', h) := Fin.to_nat x in n2f (hf _ h).

```

Ltac $\text{break_dec } H :=$

```

let H' := fresh "H" in
destruct le_lt_dec as [H'|H'];
|elim (Lt.le_not_lt _ _ H' H)
|try rewrite (n2f_ext H' H) in *; try clear H'.

```

Lemma $\text{extend_ok } n \ f : \text{bFun } n \ (\text{@extend } n \ f)$.

Lemma $\text{extend_f2n } n \ f \ (x : \text{Fin.t } n) : \text{extend } f \ (\text{f2n } x) = \text{f2n } (f \ x)$.

Lemma $\text{extend_n2f } n \ f \ x \ (h : x < n) : \text{n2f } (\text{extend_ok } f \ h) = f \ (\text{n2f } h)$.

Lemma $\text{restrict_f2n } n \ f \ hf \ (x : \text{Fin.t } n) :$

$\text{f2n } (\text{@restrict } n \ f \ hf \ x) = f \ (\text{f2n } x)$.

Lemma $\text{restrict_n2f } n \ f \ hf \ x \ (h : x < n) :$

$\text{@restrict } n \ f \ hf \ (\text{n2f } h) = \text{n2f } (hf \ _ \ h)$.

Lemma $\text{extend_surjective } n \ f :$

$\text{bSurjective } n \ (\text{@extend } n \ f) \leftrightarrow \text{Surjective } f$.

Lemma $\text{extend_injective } n \ f :$

$\text{bInjective } n \ (\text{@extend } n \ f) \leftrightarrow \text{Injective } f$.

Lemma $\text{restrict_surjective } n \ f \ h :$

$\text{Surjective } (\text{@restrict } n \ f \ h) \leftrightarrow \text{bSurjective } n \ f$.

Lemma $\text{restrict_injective } n \ f \ h :$

$\text{Injective } (\text{@restrict } n \ f \ h) \leftrightarrow \text{bInjective } n \ f$.

End FIN2RESTRICT .

Import Fin2Restrict .

We can now use Proof via the equivalence ...

Lemma $\text{bInjective_bSurjective } n \ (f : \text{nat} \rightarrow \text{nat}) :$

$\text{bFun } n \ f \rightarrow (\text{bInjective } n \ f \leftrightarrow \text{bSurjective } n \ f)$.

Lemma $\text{bSurjective_bBijective } n \ (f : \text{nat} \rightarrow \text{nat}) :$

$\text{bFun } n \ f \rightarrow \text{bSurjective } n \ f \rightarrow$

$\exists g, \text{bFun } n \ g \wedge \forall x, x < n \rightarrow g \ (f \ x) = x \wedge f \ (g \ x) = x$.

Chapter 41

Library **Coq.Unicode.Utf8**

Require Export Utf8_core.

Notation "x ≤ y" := (**le** x y) (**at level** 70, **no associativity**).

Notation "x ≥ y" := (**ge** x y) (**at level** 70, **no associativity**).

Chapter 42

Library Coq.Unicode.Utf8_core

```
Notation "∀ x .. y , P" := (∀ x, .. (∀ y, P) ..)
  (at level 200, x binder, y binder, right associativity) : type_scope.
Notation "∃ x .. y , P" := (∃ x, .. (∃ y, P) ..)
  (at level 200, x binder, y binder, right associativity) : type_scope.
Notation "x ∨ y" := (x ∨ y) (at level 85, right associativity) : type_scope.
Notation "x ∧ y" := (x ∧ y) (at level 80, right associativity) : type_scope.
Notation "x → y" := (x → y)
  (at level 99, y at level 200, right associativity) : type_scope.
Notation "x ↔ y" := (x ↔ y) (at level 95, no associativity) : type_scope.
Notation "¬ x" := (¬ x) (at level 75, right associativity) : type_scope.
Notation "x ≠ y" := (x ≠ y) (at level 70) : type_scope.
Notation "'λ' x .. y , t" := (fun x ⇒ .. (fun y ⇒ t) ..)
  (at level 200, x binder, y binder, right associativity).
```

Chapter 43

Library **Coq.Arith.PeanoNat**

Require Import NAXioms NProperties OrdersFacts.

Implementation of *NAXiomsSig* by *nat*

Module NAT

<: NAXIOMSIG

<: USUALDECIDABLETYPEFULL

<: ORDEREDTYPEFULL

<: TOTALORDER.

Operations over *nat* are defined in a separate module

Include COQ.INIT.NAT.

When including property functors, inline t eq zero one two lt le succ

All operations are well-defined (trivial here since eq is Leibniz)

Definition eq_equiv : Equivalence (@eq nat) := eq_equivalence.

Program Instance succ_wd : Proper (eq==>eq) S.

Program Instance pred_wd : Proper (eq==>eq) pred.

Program Instance add_wd : Proper (eq==>eq==>eq) plus.

Program Instance sub_wd : Proper (eq==>eq==>eq) minus.

Program Instance mul_wd : Proper (eq==>eq==>eq) mult.

Program Instance pow_wd : Proper (eq==>eq==>eq) pow.

Program Instance div_wd : Proper (eq==>eq==>eq) div.

Program Instance mod_wd : Proper (eq==>eq==>eq) modulo.

Program Instance lt_wd : Proper (eq==>eq==>iff) lt.

Program Instance testbit_wd : Proper (eq==>eq==>eq) testbit.

Bi-directional induction.

Theorem bi_induction :

$\forall A : \text{nat} \rightarrow \text{Prop}, \text{Proper } (eq ==> iff) A \rightarrow$
 $A\ 0 \rightarrow (\forall n : \text{nat}, A\ n \leftrightarrow A\ (S\ n)) \rightarrow \forall n : \text{nat}, A\ n.$

Recursion function

Definition recursion {A} : A → (nat → A → A) → nat → A :=

```

nat_rect (fun _ => A).
Instance recursion_wd {A} (Aeq : relation A) :
  Proper (Aeq ==> (eq==>Aeq==>Aeq) ==> eq ==> Aeq) recursion.
Theorem recursion_0 :
  ∀ {A} (a : A) (f : nat → A → A), recursion a f 0 = a.
Theorem recursion_succ :
  ∀ {A} (Aeq : relation A) (a : A) (f : nat → A → A),
  Aeq a a → Proper (eq==>Aeq==>Aeq) f →
  ∀ n : nat, Aeq (recursion a f (S n)) (f n (recursion a f n)).

```

43.0.1 Remaining constants not defined in Coq.Init.Nat

NB: Aliasing *le* is mandatory, since only a Definition can implement an interface Parameter...

```

Definition eq := @Logic.eq nat.
Definition le := Peano.le.
Definition lt := Peano.lt.

```

43.0.2 Basic specifications : pred add sub mul

```

Lemma pred_succ n : pred (S n) = n.
Lemma pred_0 : pred 0 = 0.
Lemma one_succ : 1 = S 0.
Lemma two_succ : 2 = S 1.
Lemma add_0_l n : 0 + n = n.
Lemma add_succ_l n m : (S n) + m = S (n + m).
Lemma sub_0_r n : n - 0 = n.
Lemma sub_succ_r n m : n - (S m) = pred (n - m).
Lemma mul_0_l n : 0 × n = 0.
Lemma mul_succ_l n m : S n × m = n × m + m.
Lemma lt_succ_r n m : n < S m ↔ n ≤ m.

```

43.0.3 Boolean comparisons

```

Lemma eqb_eq n m : eqb n m = true ↔ n = m.
Lemma leb_le n m : (n <=? m) = true ↔ n ≤ m.
Lemma ltb_lt n m : (n <? m) = true ↔ n < m.

```

43.0.4 Decidability of equality over *nat*.

```

Lemma eq_dec : ∀ n m : nat, {n = m} + {n ≠ m}.

```


43.0.5 Ternary comparison

With *nat*, it would be easier to prove first *compare_spec*, then the properties below. But then we wouldn't be able to benefit from functor *BoolOrderFacts*

Lemma *compare_eq_iff* $n\ m : (n\ ?=\ m) = \text{Eq} \leftrightarrow n = m$.

Lemma *compare_lt_iff* $n\ m : (n\ ?=\ m) = \text{Lt} \leftrightarrow n < m$.

Lemma *compare_le_iff* $n\ m : (n\ ?=\ m) \neq \text{Gt} \leftrightarrow n \leq m$.

Lemma *compare_antisym* $n\ m : (m\ ?=\ n) = \text{CompOpp}\ (n\ ?=\ m)$.

Lemma *compare_succ* $n\ m : (\text{S } n\ ?=\ \text{S } m) = (n\ ?=\ m)$.

43.0.6 Minimum, maximum

Lemma *max_l* $\forall\ n\ m, m \leq n \rightarrow \text{max } n\ m = n$.

Lemma *max_r* $\forall\ n\ m, n \leq m \rightarrow \text{max } n\ m = m$.

Lemma *min_l* $\forall\ n\ m, n \leq m \rightarrow \text{min } n\ m = n$.

Lemma *min_r* $\forall\ n\ m, m \leq n \rightarrow \text{min } n\ m = m$.

Some more advanced properties of comparison and orders, including *compare_spec* and *lt_irrefl* and *lt_eq_cases*.

Include *BOOLORDERFACTS*.

We can now derive all properties of basic functions and orders, and use these properties for proving the specs of more advanced functions.

Include *NBASICPROP* <+ *USUALMINMAXLOGICALPROPERTIES* <+ *USUALMINMAXDECPROPERTIES*.

43.0.7 Power

Lemma *pow_neg_r* $a\ b : b < 0 \rightarrow a^b = 0$.

Lemma *pow_0_r* $a : a^0 = 1$.

Lemma *pow_succ_r* $a\ b : 0 \leq b \rightarrow a^{(\text{S } b)} = a \times a^b$.

43.0.8 Square

Lemma *square_spec* $n : \text{square } n = n \times n$.

43.0.9 Parity

Definition *Even* $n := \exists\ m, n = 2 \times m$.

Definition *Odd* $n := \exists\ m, n = 2 \times m + 1$.

Module *PRIVATE_PARITY*.

Lemma *Even_1* $:\neg \text{Even } 1$.

Lemma *Even_2* $n : \text{Even } n \leftrightarrow \text{Even } (\text{S } (\text{S } n))$.

Lemma Odd_0 : \neg Odd 0.
 Lemma Odd_2 n : Odd n \leftrightarrow Odd (S (S n)).
 End PRIVATE_PARITY.
 Import Private_Parity.
 Lemma even_spec : \forall n, even n = true \leftrightarrow Even n.
 Lemma odd_spec : \forall n, odd n = true \leftrightarrow Odd n.

43.0.10 Division

Lemma divmod_spec : \forall x y q u, $u \leq y \rightarrow$
 let (q', u') := divmod x y q u in
 $x + (S y) * q + (y - u) = (S y) * q' + (y - u') \wedge u' \leq y$.
 Lemma div_mod x y : $y \neq 0 \rightarrow x = y * (x / y) + x \bmod y$.
 Lemma mod_bound_pos x y : $0 \leq x \rightarrow 0 < y \rightarrow 0 \leq x \bmod y < y$.

43.0.11 Square root

Lemma sqrt_iter_spec : \forall k p q r,
 $q = p + p \rightarrow r \leq q \rightarrow$
 let s := sqrt_iter k p q r in
 $s * s \leq k + p * p + (q - r) < (S s) * (S s)$.
 Lemma sqrt_specif n : $(\text{sqrt } n) * (\text{sqrt } n) \leq n < S (\text{sqrt } n) * S (\text{sqrt } n)$.
 Definition sqrt_spec a (Ha : $0 \leq a$) := sqrt_specif a.
 Lemma sqrt_neg a : $a < 0 \rightarrow \text{sqrt } a = 0$.

43.0.12 Logarithm

Lemma log2_iter_spec : \forall k p q r,
 $2^{(S p)} = q + S r \rightarrow r < 2^p \rightarrow$
 let s := log2_iter k p q r in
 $2^s \leq k + q < 2^{(S s)}$.
 Lemma log2_spec n : $0 < n \rightarrow$
 $2^{(\text{log2 } n)} \leq n < 2^{(S (\text{log2 } n))}$.
 Lemma log2_nonpos n : $n \leq 0 \rightarrow \text{log2 } n = 0$.

43.0.13 Gcd

Definition divide x y := $\exists z, y = z * x$.
 Notation "(x | y)" := (divide x y) (at level 0) : nat_scope.
 Lemma gcd_divide : $\forall a b, (\text{gcd } a b \mid a) \wedge (\text{gcd } a b \mid b)$.
 Lemma gcd_divide_l : $\forall a b, (\text{gcd } a b \mid a)$.

Lemma gcd_divide_r : $\forall a b, (\text{gcd } a b \mid b)$.

Lemma gcd_greatest : $\forall a b c, (c \mid a) \rightarrow (c \mid b) \rightarrow (c \mid \text{gcd } a b)$.

Lemma gcd_nonneg a b : $0 \leq \text{gcd } a b$.

43.0.14 Bitwise operations

Lemma div2_double n : $\text{div2 } (2 \times n) = n$.

Lemma div2_succ_double n : $\text{div2 } (\text{S } (2 \times n)) = n$.

Lemma le_div2 n : $\text{div2 } (\text{S } n) \leq n$.

Lemma lt_div2 n : $0 < n \rightarrow \text{div2 } n < n$.

Lemma div2_decr a n : $a \leq \text{S } n \rightarrow \text{div2 } a \leq n$.

Lemma double_twice : $\forall n, \text{double } n = 2 \times n$.

Lemma testbit_0_l : $\forall n, \text{testbit } 0 n = \text{false}$.

Lemma testbit_odd_0 a : $\text{testbit } (2 \times a + 1) 0 = \text{true}$.

Lemma testbit_even_0 a : $\text{testbit } (2 \times a) 0 = \text{false}$.

Lemma testbit_odd_succ' a n : $\text{testbit } (2 \times a + 1) (\text{S } n) = \text{testbit } a n$.

Lemma testbit_even_succ' a n : $\text{testbit } (2 \times a) (\text{S } n) = \text{testbit } a n$.

Lemma shiftr_specif : $\forall a n m,$
 $\text{testbit } (\text{shiftr } a n) m = \text{testbit } a (m + n)$.

Lemma shiftr_specif_high : $\forall a n m, n \leq m \rightarrow$
 $\text{testbit } (\text{shiftr } a n) m = \text{testbit } a (m - n)$.

Lemma shiftr_spec_low : $\forall a n m, m < n \rightarrow$
 $\text{testbit } (\text{shiftr } a n) m = \text{false}$.

Lemma div2_bitwise : $\forall op n a b,$
 $\text{div2 } (\text{bitwise } op (\text{S } n) a b) = \text{bitwise } op n (\text{div2 } a) (\text{div2 } b)$.

Lemma odd_bitwise : $\forall op n a b,$
 $\text{odd } (\text{bitwise } op (\text{S } n) a b) = op (\text{odd } a) (\text{odd } b)$.

Lemma testbit_bitwise_1 : $\forall op, (\forall b, op \text{ false } b = \text{false}) \rightarrow$
 $\forall n m a b, a \leq n \rightarrow$
 $\text{testbit } (\text{bitwise } op n a b) m = op (\text{testbit } a m) (\text{testbit } b m)$.

Lemma testbit_bitwise_2 : $\forall op, op \text{ false } \text{false} = \text{false} \rightarrow$
 $\forall n m a b, a \leq n \rightarrow b \leq n \rightarrow$
 $\text{testbit } (\text{bitwise } op n a b) m = op (\text{testbit } a m) (\text{testbit } b m)$.

Lemma land_spec a b n :
 $\text{testbit } (\text{land } a b) n = \text{testbit } a n \ \&\& \ \text{testbit } b n$.

Lemma ldif_spec a b n :
 $\text{testbit } (\text{ldif } a b) n = \text{testbit } a n \ \&\& \ \text{negb } (\text{testbit } b n)$.

Lemma lor_spec a b n :
 $\text{testbit } (\text{lor } a b) n = \text{testbit } a n \ || \ \text{testbit } b n$.

Lemma `lxor_spec a b n :`
`testbit (lxor a b) n = xorb (testbit a n) (testbit b n).`

Lemma `div2_spec a : div2 a = shiftr a 1.`

Aliases with extra dummy hypothesis, to fulfil the interface

Definition `testbit_odd_succ a n (_:0 ≤ n) := testbit_odd_succ' a n.`

Definition `testbit_even_succ a n (_:0 ≤ n) := testbit_even_succ' a n.`

Lemma `testbit_neg_r a n (H:n<0) : testbit a n = false.`

Definition `shiftr_spec_high a n m (_:0 ≤ m) := shiftr_specif_high a n m.`

Definition `shiftr_spec a n m (_:0 ≤ m) := shiftr_specif a n m.`

Properties of advanced functions (pow, sqrt, log2, ...)

Include `NEXTRAPROP.`

End `NAT.`

Re-export notations that should be available even when the *Nat* module is not imported.

Infix `"^" := Nat.pow : nat_scope.`

Infix `"=" := Nat.eqb (at level 70) : nat_scope.`

Infix `"<=" := Nat.leb (at level 70) : nat_scope.`

Infix `"<" := Nat.ltb (at level 70) : nat_scope.`

Infix `"?=" := Nat.compare (at level 70) : nat_scope.`

Infix `"/" := Nat.div : nat_scope.`

Infix `"mod" := Nat.modulo (at level 40, no associativity) : nat_scope.`

Hint `Unfold Nat.le : core.`

Hint `Unfold Nat.lt : core.`

Nat contains an *order* tactic for natural numbers

Note that *Nat.order* is domain-agnostic: it will not prove $1 \leq 2$ or $x \leq x+x$, but rather things like $x \leq y \rightarrow y \leq x \rightarrow x=y$.

Section `TestOrder.`

Let *test* : $\forall x y, x \leq y \rightarrow y \leq x \rightarrow x=y$.

End `TestOrder.`

Chapter 44

Library `Coq.Arith.Arith_base`

```
Require Export PeanoNat.  
Require Export Le.  
Require Export Lt.  
Require Export Plus.  
Require Export Gt.  
Require Export Minus.  
Require Export Mult.  
Require Export Between.  
Require Export Peano_dec.  
Require Export Compare_dec.  
Require Export Factorial.  
Require Export EqNat.  
Require Export Wf_nat.
```

Chapter 45

Library **Coq.Arith.Arith**

```
Require Export Arith_base.  
Require Export ArithRing.
```

Chapter 46

Library **Coq.Arith.Between**

```
Require Import Le.
Require Import Lt.

Local Open Scope nat_scope.

Implicit Types k l p q r : nat.

Section Between.
  Variables P Q : nat → Prop.

  Inductive between k : nat → Prop :=
    | bet_emp : between k k
    | bet_S : ∀ l, between k l → P l → between k (S l).

  Hint Constructors between: arith v62.

  Lemma bet_eq : ∀ k l, l = k → between k l.

  Hint Resolve bet_eq: arith v62.

  Lemma between_le : ∀ k l, between k l → k ≤ l.
  Hint Immediate between_le: arith v62.

  Lemma between_Sk_l : ∀ k l, between k l → S k ≤ l → between (S k) l.
  Hint Resolve between_Sk_l: arith v62.

  Lemma between_restr :
    ∀ k l (m:nat), k ≤ l → l ≤ m → between k m → between l m.

  Inductive exists_between k : nat → Prop :=
    | exists_S : ∀ l, exists_between k l → exists_between k (S l)
    | exists_le : ∀ l, k ≤ l → Q l → exists_between k (S l).

  Hint Constructors exists_between: arith v62.

  Lemma exists_le_S : ∀ k l, exists_between k l → S k ≤ l.

  Lemma exists_lt : ∀ k l, exists_between k l → k < l.
  Hint Immediate exists_le_S exists_lt: arith v62.

  Lemma exists_S_le : ∀ k l, exists_between k (S l) → k ≤ l.
  Hint Immediate exists_S_le: arith v62.
```

Definition `in_int` $p\ q\ r := p \leq r \wedge r < q$.
Lemma `in_int_intro` : $\forall p\ q\ r, p \leq r \rightarrow r < q \rightarrow \text{in_int}\ p\ q\ r$.
Hint `Resolve in_int_intro`: *arith v62*.
Lemma `in_int_lt` : $\forall p\ q\ r, \text{in_int}\ p\ q\ r \rightarrow p < q$.
Lemma `in_int_p_Sq` :
 $\forall p\ q\ r, \text{in_int}\ p\ (\text{S } q)\ r \rightarrow \text{in_int}\ p\ q\ r \vee r = q$:>**nat**.
Lemma `in_int_S` : $\forall p\ q\ r, \text{in_int}\ p\ q\ r \rightarrow \text{in_int}\ p\ (\text{S } q)\ r$.
Hint `Resolve in_int_S`: *arith v62*.
Lemma `in_int_Sp_q` : $\forall p\ q\ r, \text{in_int}\ (\text{S } p)\ q\ r \rightarrow \text{in_int}\ p\ q\ r$.
Hint `Immediate in_int_Sp_q`: *arith v62*.
Lemma `between_in_int` :
 $\forall k\ l, \text{between}\ k\ l \rightarrow \forall r, \text{in_int}\ k\ l\ r \rightarrow P\ r$.
Lemma `in_int_between` :
 $\forall k\ l, k \leq l \rightarrow (\forall r, \text{in_int}\ k\ l\ r \rightarrow P\ r) \rightarrow \text{between}\ k\ l$.
Lemma `exists_in_int` :
 $\forall k\ l, \text{exists_between}\ k\ l \rightarrow \text{exists2}\ m : \text{nat}, \text{in_int}\ k\ l\ m \ \&\ Q\ m$.
Lemma `in_int_exists` : $\forall k\ l\ r, \text{in_int}\ k\ l\ r \rightarrow Q\ r \rightarrow \text{exists_between}\ k\ l$.
Lemma `between_or_exists` :
 $\forall k\ l,$
 $k \leq l \rightarrow$
 $(\forall n:\text{nat}, \text{in_int}\ k\ l\ n \rightarrow P\ n \vee Q\ n) \rightarrow$
 $\text{between}\ k\ l \vee \text{exists_between}\ k\ l$.
Lemma `between_not_exists` :
 $\forall k\ l,$
 $\text{between}\ k\ l \rightarrow$
 $(\forall n:\text{nat}, \text{in_int}\ k\ l\ n \rightarrow P\ n \rightarrow \neg Q\ n) \rightarrow \neg \text{exists_between}\ k\ l$.
Inductive `P_nth` (*init*:**nat**) : **nat** \rightarrow **nat** \rightarrow **Prop** :=
| `nth_O` : `P_nth` *init* *init* 0
| `nth_S` :
 $\forall k\ l\ (n:\text{nat}),$
 $\text{P_nth}\ \text{init}\ k\ n \rightarrow \text{between}\ (\text{S } k)\ l \rightarrow Q\ l \rightarrow \text{P_nth}\ \text{init}\ l\ (\text{S } n)$.
Lemma `nth_le` : $\forall (init:\text{nat})\ l\ (n:\text{nat}), \text{P_nth}\ \text{init}\ l\ n \rightarrow \text{init} \leq l$.
Definition `eventually` (*n*:**nat**) := `exists2` $k : \text{nat}, k \leq n \ \&\ Q\ k$.
Lemma `event_O` : `eventually` 0 $\rightarrow Q\ 0$.
End `Between`.
Hint `Resolve nth_O bet_S bet_emp bet_eq between_Sk_l exists_S exists_le in_int_S in_int_intro`: *arith v62*.
Hint `Immediate in_int_Sp_q exists_le_S exists_S_le`: *arith v62*.

Chapter 47

Library **Coq.Arith.Bool_nat**

```
Require Export Compare_dec.  
Require Export Peano_dec.  
Require Import Sumbool.  
Local Open Scope nat_scope.  
Implicit Types m n x y : nat.
```

The decidability of equality and order relations over type *nat* give some boolean functions with the adequate specification.

```
Definition notzerop n := sumbool_not _ _ (zerop n).  
Definition lt_ge_dec :  $\forall x y, \{x < y\} + \{x \geq y\}$  :=  
  fun n m  $\Rightarrow$  sumbool_not _ _ (le_lt_dec m n).  
Definition nat_lt_ge_bool x y := bool_of_sumbool (lt_ge_dec x y).  
Definition nat_ge_lt_bool x y :=  
  bool_of_sumbool (sumbool_not _ _ (lt_ge_dec x y)).  
Definition nat_le_gt_bool x y := bool_of_sumbool (le_gt_dec x y).  
Definition nat_gt_le_bool x y :=  
  bool_of_sumbool (sumbool_not _ _ (le_gt_dec x y)).  
Definition nat_eq_bool x y := bool_of_sumbool (eq_nat_dec x y).  
Definition nat_noteq_bool x y :=  
  bool_of_sumbool (sumbool_not _ _ (eq_nat_dec x y)).  
Definition zerop_bool x := bool_of_sumbool (zerop x).  
Definition notzerop_bool x := bool_of_sumbool (notzerop x).
```

Chapter 48

Library `Coq.Arith.Compare_dec`

```
Require Import Le Lt Gt Decidable PeanoNat.
Local Open Scope nat_scope.
Implicit Types m n x y : nat.
Definition zerop n : {n = 0} + {0 < n}.
Definition lt_eq_lt_dec n m : {n < m} + {n = m} + {m < n}.
Definition gt_eq_gt_dec n m : {m > n} + {n = m} + {n > m}.
Definition le_lt_dec n m : {n ≤ m} + {m < n}.
Definition le_le_S_dec n m : {n ≤ m} + {S m ≤ n}.
Definition le_ge_dec n m : {n ≤ m} + {n ≥ m}.
Definition le_gt_dec n m : {n ≤ m} + {n > m}.
Definition le_lt_eq_dec n m : n ≤ m → {n < m} + {n = m}.
Theorem le_dec n m : {n ≤ m} + {¬ n ≤ m}.
Theorem lt_dec n m : {n < m} + {¬ n < m}.
Theorem gt_dec n m : {n > m} + {¬ n > m}.
Theorem ge_dec n m : {n ≥ m} + {¬ n ≥ m}.
```

Proofs of decidability

```
Theorem dec_le n m : decidable (n ≤ m).
Theorem dec_lt n m : decidable (n < m).
Theorem dec_gt n m : decidable (n > m).
Theorem dec_ge n m : decidable (n ≥ m).
Theorem not_eq n m : n ≠ m → n < m ∨ m < n.
Theorem not_le n m : ¬ n ≤ m → n > m.
Theorem not_gt n m : ¬ n > m → n ≤ m.
Theorem not_ge n m : ¬ n ≥ m → n < m.
Theorem not_lt n m : ¬ n < m → n ≥ m.
```

A ternary comparison function in the spirit of *Z.compare*. See now *Nat.compare* and its properties. In scope *nat_scope*, the notation for *Nat.compare* is “*?=*”

Notation *nat_compare* := *Nat.compare* (*compat* "8.4").

Notation *nat_compare_spec* := *Nat.compare_spec* (*compat* "8.4").

Notation *nat_compare_eq_iff* := *Nat.compare_eq_iff* (*compat* "8.4").

Notation *nat_compare_S* := *Nat.compare_succ* (*compat* "8.4").

Lemma *nat_compare_lt* *n m* : $n < m \leftrightarrow (n \text{ ?= } m) = \text{Lt}$.

Lemma *nat_compare_gt* *n m* : $n > m \leftrightarrow (n \text{ ?= } m) = \text{Gt}$.

Lemma *nat_compare_le* *n m* : $n \leq m \leftrightarrow (n \text{ ?= } m) \neq \text{Gt}$.

Lemma *nat_compare_ge* *n m* : $n \geq m \leftrightarrow (n \text{ ?= } m) \neq \text{Lt}$.

Some projections of the above equivalences.

Lemma *nat_compare_eq* *n m* : $(n \text{ ?= } m) = \text{Eq} \rightarrow n = m$.

Lemma *nat_compare_Lt_Lt* *n m* : $(n \text{ ?= } m) = \text{Lt} \rightarrow n < m$.

Lemma *nat_compare_Gt_gt* *n m* : $(n \text{ ?= } m) = \text{Gt} \rightarrow n > m$.

A previous definition of *nat_compare* in terms of *lt_eq_lt_dec*. The new version avoids the creation of proof parts.

Definition *nat_compare_alt* (*n m*:*nat*) :=

```
match lt_eq_lt_dec n m with
| inleft (left _) => Lt
| inleft (right _) => Eq
| inright _ => Gt
end.
```

Lemma *nat_compare_equiv* *n m* : $(n \text{ ?= } m) = \text{nat_compare_alt } n \text{ } m$.

A boolean version of *le* over *nat*. See now *Nat.leb* and its properties. In scope *nat_scope*, the notation for *Nat.leb* is “*<=?*”

Notation *leb* := *Nat.leb* (*compat* "8.4").

Notation *leb_iff* := *Nat.leb_le* (*compat* "8.4").

Lemma *leb_iff_conv* *m n* : $(n <=? m) = \text{false} \leftrightarrow m < n$.

Lemma *leb_correct* *m n* : $m \leq n \rightarrow (m <=? n) = \text{true}$.

Lemma *leb_complete* *m n* : $(m <=? n) = \text{true} \rightarrow m \leq n$.

Lemma *leb_correct_conv* *m n* : $m < n \rightarrow (n <=? m) = \text{false}$.

Lemma *leb_complete_conv* *m n* : $(n <=? m) = \text{false} \rightarrow m < n$.

Lemma *leb_compare* *n m* : $(n <=? m) = \text{true} \leftrightarrow (n \text{ ?= } m) \neq \text{Gt}$.

Chapter 49

Library Coq.Arith.Compare

Equality is decidable on *nat*

Local Open Scope *nat_scope*.

Notation *not_eq_sym* := *not_eq_sym* (*only parsing*).

Implicit Types *m n p q* : *nat*.

Require Import Arith_base.

Require Import Peano_dec.

Require Import Compare_dec.

Definition *le_or_le_S* := *le_le_S_dec*.

Definition *Pcompare* := *gt_eq_gt_dec*.

Lemma *le_dec* : $\forall n\ m, \{n \leq m\} + \{m \leq n\}$.

Definition *lt_or_eq* *n m* := $\{m > n\} + \{n = m\}$.

Lemma *le_decide* : $\forall n\ m, n \leq m \rightarrow \text{lt_or_eq } n\ m$.

Lemma *le_le_S_eq* : $\forall n\ m, n \leq m \rightarrow S\ n \leq m \vee n = m$.

Lemma *discrete_nat* :

$\forall n\ m, n < m \rightarrow S\ n = m \vee (\exists r : \text{nat}, m = S\ (S\ (n + r)))$.

Require Export Wf_nat.

Require Export Min Max.

Chapter 50

Library Coq.Arith.Div2

Nota : this file is OBSOLETE, and left only for compatibility. Please consider using *Nat.div2* directly, and results about it (see file PeanoNat).

Require Import PeanoNat Even.

Local Open Scope nat_scope.

Implicit Type n : nat.

Here we define $n/2$ and prove some of its properties

Notation $\text{div2} := \text{Nat.div2}$ (*compat* "8.4").

Since *div2* is recursively defined on 0, 1 and $(S (S n))$, it is useful to prove the corresponding induction principle

Lemma ind_0_1_SS :

$\forall P : \text{nat} \rightarrow \text{Prop},$

$P\ 0 \rightarrow P\ 1 \rightarrow (\forall n, P\ n \rightarrow P\ (S\ (S\ n))) \rightarrow \forall n, P\ n.$

$0 < n \Rightarrow n/2 < n$

Lemma lt_div2 $n : 0 < n \rightarrow \text{div2}\ n < n.$

Hint Resolve lt_div2: arith.

Properties related to the parity

Lemma even_div2 $n : \text{even}\ n \rightarrow \text{div2}\ n = \text{div2}\ (S\ n).$

Lemma odd_div2 $n : \text{odd}\ n \rightarrow S\ (\text{div2}\ n) = \text{div2}\ (S\ n).$

Lemma div2_even $n : \text{div2}\ n = \text{div2}\ (S\ n) \rightarrow \text{even}\ n.$

Lemma div2_odd $n : S\ (\text{div2}\ n) = \text{div2}\ (S\ n) \rightarrow \text{odd}\ n.$

Hint Resolve even_div2 div2_even odd_div2 div2_odd: arith.

Lemma even_odd_div2 $n :$

$(\text{even}\ n \leftrightarrow \text{div2}\ n = \text{div2}\ (S\ n)) \wedge$

$(\text{odd}\ n \leftrightarrow S\ (\text{div2}\ n) = \text{div2}\ (S\ n)).$

Properties related to the double $(2n)$

Notation double $:= \text{Nat.double}$ (*compat* "8.4").

Hint Unfold double Nat.double: *arith*.

Lemma double_S n : double (S n) = S (S (double n)).

Lemma double_plus n m : double ($n + m$) = double n + double m .

Hint Resolve double_S: *arith*.

Lemma even_odd_double n :

(even $n \leftrightarrow n = \text{double} (\text{div2 } n)$) \wedge (odd $n \leftrightarrow n = \text{S} (\text{double} (\text{div2 } n))$).

Specializations

Lemma even_double n : even $n \rightarrow n = \text{double} (\text{div2 } n)$.

Lemma double_even n : $n = \text{double} (\text{div2 } n) \rightarrow \text{even } n$.

Lemma odd_double n : odd $n \rightarrow n = \text{S} (\text{double} (\text{div2 } n))$.

Lemma double_odd n : $n = \text{S} (\text{double} (\text{div2 } n)) \rightarrow \text{odd } n$.

Hint Resolve even_double double_even odd_double double_odd: *arith*.

Application:

- if n is even then there is a p such that $n = 2p$
- if n is odd then there is a p such that $n = 2p+1$

(Immediate: it is $n/2$)

Lemma even_2n : $\forall n, \text{even } n \rightarrow \{p : \text{nat} \mid n = \text{double } p\}$.

Lemma odd_S2n : $\forall n, \text{odd } n \rightarrow \{p : \text{nat} \mid n = \text{S} (\text{double } p)\}$.

Doubling before dividing by two brings back to the initial number.

Lemma div2_double n : div2 ($2 \times n$) = n .

Lemma div2_double_plus_one n : div2 (S ($2 \times n$)) = n .

Chapter 51

Library **Coq.Arith.EqNat**

```
Require Import PeanoNat.  
Local Open Scope nat_scope.
```

Equality on natural numbers

51.1 Propositional equality

```
Fixpoint eq_nat n m : Prop :=  
  match n, m with  
  | O, O => True  
  | O, S _ => False  
  | S _, O => False  
  | S n1, S m1 => eq_nat n1 m1  
end.
```

Theorem `eq_nat_refl` n : `eq_nat` n n .

Hint `Resolve` `eq_nat_refl`: *arith v62*.

eq restricted to *nat* and *eq_nat* are equivalent

Theorem `eq_nat_is_eq` n m : `eq_nat` n m \leftrightarrow $n = m$.

Lemma `eq_eq_nat` n m : $n = m \rightarrow$ `eq_nat` n m .

Lemma `eq_nat_eq` n m : `eq_nat` n $m \rightarrow$ $n = m$.

Hint `Immediate` `eq_eq_nat` `eq_nat_eq`: *arith v62*.

Theorem `eq_nat_elim` :

$\forall n (P:\text{nat} \rightarrow \text{Prop}), P\ n \rightarrow \forall m, \text{eq_nat}\ n\ m \rightarrow P\ m$.

Theorem `eq_nat_decide` : $\forall n\ m, \{\text{eq_nat}\ n\ m\} + \{\neg \text{eq_nat}\ n\ m\}$.

51.2 Boolean equality on *nat*.

We reuse the one already defined in module *Nat*. In scope *nat_scope*, the notation “=” can be used.

```

Notation beq_nat := Nat.eqb (compat "8.4").
Notation beq_nat_true_iff := Nat.eqb_eq (compat "8.4").
Notation beq_nat_false_iff := Nat.eqb_neq (compat "8.4").
Lemma beq_nat_refl n : true = (n =? n).
Lemma beq_nat_true n m : (n =? m) = true → n=m.
Lemma beq_nat_false n m : (n =? m) = false → n≠m.

```

TODO: is it really useful here to have a Defined ? Otherwise we could use Nat.eqb_eq

```

Definition beq_nat_eq : ∀ n m, true = (n =? m) → n = m.

```


Chapter 52

Library Coq.Arith.Euclid

```
Require Import Mult.
Require Import Compare_dec.
Require Import Wf_nat.

Local Open Scope nat_scope.

Implicit Types a b n q r : nat.

Inductive diveucl a b : Set :=
  divex :  $\forall q\ r, b > r \rightarrow a = q \times b + r \rightarrow \text{diveucl } a\ b$ .

Lemma eucl_dev :  $\forall n, n > 0 \rightarrow \forall m:\text{nat}, \text{diveucl } m\ n$ .

Lemma quotient :
   $\forall n,$ 
   $n > 0 \rightarrow$ 
   $\forall m:\text{nat}, \{q : \text{nat} \mid \exists r : \text{nat}, m = q \times n + r \wedge n > r\}$ .

Lemma modulo :
   $\forall n,$ 
   $n > 0 \rightarrow$ 
   $\forall m:\text{nat}, \{r : \text{nat} \mid \exists q : \text{nat}, m = q \times n + r \wedge n > r\}$ .
```

Chapter 53

Library **Coq.Arith.Even**

Nota : this file is OBSOLETE, and left only for compatibility. Please consider instead predicates *Nat.Even* and *Nat.Odd* and Boolean functions *Nat.even* and *Nat.odd*.

Here we define the predicates *even* and *odd* by mutual induction and we prove the decidability and the exclusion of those predicates. The main results about parity are proved in the module Div2.

```
Require Import PeanoNat.  
Local Open Scope nat_scope.  
Implicit Types m n : nat.
```

53.1 Inductive definition of *even* and *odd*

```
Inductive even : nat → Prop :=  
  | even_O : even 0  
  | even_S : ∀ n, odd n → even (S n)  
with odd : nat → Prop :=  
  odd_S : ∀ n, even n → odd (S n).  
Hint Constructors even: arith.  
Hint Constructors odd: arith.
```

53.2 Equivalence with predicates *Nat.Even* and *Nat.odd*

```
Lemma even_equiv : ∀ n, even n ↔ Nat.Even n.  
Lemma odd_equiv : ∀ n, odd n ↔ Nat.Odd n.  
Basic facts  
Lemma even_or_odd n : even n ∨ odd n.  
Lemma even_odd_dec n : {even n} + {odd n}.  
Lemma not_even_and_odd n : even n → odd n → False.
```

53.3 Facts about *even* & *odd* wrt. *plus*

```

Ltac parity2bool :=
  rewrite ?even_equiv, ?odd_equiv, ← ?Nat.even_spec, ← ?Nat.odd_spec.

Ltac parity_binop_spec :=
  rewrite ?Nat.even_add, ?Nat.odd_add, ?Nat.even_mul, ?Nat.odd_mul.

Ltac parity_binop :=
  parity2bool; parity_binop_spec; unfold Nat.odd;
  do 2 destruct Nat.even; simpl; tauto.

Lemma even_plus_split n m :
  even (n + m) → even n ∧ even m ∨ odd n ∧ odd m.

Lemma odd_plus_split n m :
  odd (n + m) → odd n ∧ even m ∨ even n ∧ odd m.

Lemma even_even_plus n m : even n → even m → even (n + m).

Lemma odd_plus_l n m : odd n → even m → odd (n + m).

Lemma odd_plus_r n m : even n → odd m → odd (n + m).

Lemma odd_even_plus n m : odd n → odd m → even (n + m).

Lemma even_plus_aux n m :
  (odd (n + m) ↔ odd n ∧ even m ∨ even n ∧ odd m) ∧
  (even (n + m) ↔ even n ∧ even m ∨ odd n ∧ odd m).

Lemma even_plus_even_inv_r n m : even (n + m) → even n → even m.

Lemma even_plus_even_inv_l n m : even (n + m) → even m → even n.

Lemma even_plus_odd_inv_r n m : even (n + m) → odd n → odd m.

Lemma even_plus_odd_inv_l n m : even (n + m) → odd m → odd n.

Lemma odd_plus_even_inv_l n m : odd (n + m) → odd m → even n.

Lemma odd_plus_even_inv_r n m : odd (n + m) → odd n → even m.

Lemma odd_plus_odd_inv_l n m : odd (n + m) → even m → odd n.

Lemma odd_plus_odd_inv_r n m : odd (n + m) → even n → odd m.

```

53.4 Facts about *even* and *odd* wrt. *mult*

```

Lemma even_mult_aux n m :
  (odd (n × m) ↔ odd n ∧ odd m) ∧ (even (n × m) ↔ even n ∨ even m).

Lemma even_mult_l n m : even n → even (n × m).

Lemma even_mult_r n m : even m → even (n × m).

Lemma even_mult_inv_r n m : even (n × m) → odd n → even m.

Lemma even_mult_inv_l n m : even (n × m) → odd m → even n.

Lemma odd_mult n m : odd n → odd m → odd (n × m).

```

Lemma odd_mult_inv_l $n\ m$: odd $(n \times m) \rightarrow$ odd n .

Lemma odd_mult_inv_r $n\ m$: odd $(n \times m) \rightarrow$ odd m .

Hint Resolve

*even_even_plus odd_even_plus odd_plus_l odd_plus-r
even_mult_l even_mult_r even_mult_l even_mult_r odd_mult : arith.*

Chapter 54

Library **Coq.Arith.Factorial**

```
Require Import PeanoNat Plus Mult Lt.
```

```
Local Open Scope nat_scope.
```

```
Factorial
```

```
Fixpoint fact (n:nat) : nat :=
```

```
  match n with
```

```
    | O => 1
```

```
    | S n => S n × fact n
```

```
  end.
```

```
Lemma lt_O_fact n : 0 < fact n.
```

```
Lemma fact_neq_0 n : fact n ≠ 0.
```

```
Lemma fact_le n m : n ≤ m → fact n ≤ fact m.
```

Chapter 55

Library Coq.Arith.Gt

Theorems about *gt* in *nat*.

This file is DEPRECATED now, see module *PeanoNat.Nat* instead, which favor *lt* over *gt*.
gt is defined in *Init/Peano.v* as:

Definition gt (n m:nat) := m < n.

Require Import PeanoNat Le Lt Plus.

Local Open Scope nat_scope.

55.1 Order and successor

Theorem gt_Sn_O n : S n > 0.

Theorem gt_Sn_n n : S n > n.

Theorem gt_n_S n m : n > m → S n > S m.

Lemma gt_S_n n m : S m > S n → m > n.

Theorem gt_S n m : S n > m → n > m ∨ m = n.

Lemma gt_pred n m : m > S n → pred m > n.

55.2 Irreflexivity

Lemma gt_irrefl n : ¬ n > n.

55.3 Asymmetry

Lemma gt_asym n m : n > m → ¬ m > n.

55.4 Relating strict and large orders

Lemma `le_not_gt` $n\ m : n \leq m \rightarrow \neg n > m$.

Lemma `gt_not_le` $n\ m : n > m \rightarrow \neg n \leq m$.

Theorem `le_S_gt` $n\ m : \mathbf{S}\ n \leq m \rightarrow m > n$.

Lemma `gt_S_le` $n\ m : \mathbf{S}\ m > n \rightarrow n \leq m$.

Lemma `gt_le_S` $n\ m : m > n \rightarrow \mathbf{S}\ n \leq m$.

Lemma `le_gt_S` $n\ m : n \leq m \rightarrow \mathbf{S}\ m > n$.

55.5 Transitivity

Theorem `le_gt_trans` $n\ m\ p : m \leq n \rightarrow m > p \rightarrow n > p$.

Theorem `gt_le_trans` $n\ m\ p : n > m \rightarrow p \leq m \rightarrow n > p$.

Lemma `gt_trans` $n\ m\ p : n > m \rightarrow m > p \rightarrow n > p$.

Theorem `gt_trans_S` $n\ m\ p : \mathbf{S}\ n > m \rightarrow m > p \rightarrow n > p$.

55.6 Comparison to 0

Theorem `gt_0_eq` $n : n > 0 \vee 0 = n$.

55.7 Simplification and compatibility

Lemma `plus_gt_reg_l` $n\ m\ p : p + n > p + m \rightarrow n > m$.

Lemma `plus_gt_compat_l` $n\ m\ p : n > m \rightarrow p + n > p + m$.

55.8 Hints

Hint `Resolve` `gt_Sn_O` `gt_Sn_n` `gt_n_S` : *arith v62*.

Hint `Immediate` `gt_S_n` `gt_pred` : *arith v62*.

Hint `Resolve` `gt_irrefl` `gt_asym` : *arith v62*.

Hint `Resolve` `le_not_gt` `gt_not_le` : *arith v62*.

Hint `Immediate` `le_S_gt` `gt_S_le` : *arith v62*.

Hint `Resolve` `gt_le_S` `le_gt_S` : *arith v62*.

Hint `Resolve` `gt_trans_S` `le_gt_trans` `gt_le_trans` : *arith v62*.

Hint `Resolve` `plus_gt_compat_l` : *arith v62*.

Chapter 56

Library Coq.Arith.Le

Order on natural numbers.

This file is mostly OBSOLETE now, see module *PeanoNat.Nat* instead.

le is defined in *Init/Peano.v* as:

```
Inductive le (n:nat) : nat -> Prop :=
| le_n : n <= n
| le_S : forall m:nat, n <= m -> n <= S m
```

where "n <= m" := (le n m) : nat_scope.

Require Import PeanoNat.

Local Open Scope nat_scope.

56.1 *le* is an order on *nat*

Notation le_refl := Nat.le_refl (compat "8.4").

Notation le_trans := Nat.le_trans (compat "8.4").

Notation le_antisym := Nat.le_antisymm (compat "8.4").

Hint Resolve le_trans: arith v62.

Hint Immediate le_antisym: arith v62.

56.2 Properties of *le* w.r.t 0

Notation le_0_n := Nat.le_0_l (compat "8.4"). Notation le_Sn_0 := Nat.nle_succ_0 (compat "8.4").

Lemma le_n_0_eq n : $n \leq 0 \rightarrow 0 = n$.

56.3 Properties of *le* w.r.t successor

See also *Nat.succ_le_mono*.

Theorem le_n_S : $\forall n\ m, n \leq m \rightarrow S\ n \leq S\ m$.

Theorem `le_S_n` : $\forall n\ m, \mathbf{S}\ n \leq \mathbf{S}\ m \rightarrow n \leq m$.

Notation `le_n_Sn` := `Nat.le_succ_diag_r` (*compat* "8.4"). **Notation** `le_Sn_n` := `Nat.nle_succ_diag_l` (*compat* "8.4").

Theorem `le_Sn_le` : $\forall n\ m, \mathbf{S}\ n \leq m \rightarrow n \leq m$.

Hint `Resolve` `le_0_n` `le_Sn_0` : *arith v62*.

Hint `Resolve` `le_n_S` `le_n_Sn` `le_Sn_n` : *arith v62*.

Hint `Immediate` `le_n_0_eq` `le_Sn_le` `le_S_n` : *arith v62*.

56.4 Properties of *le* w.r.t predecessor

Notation `le_pred_n` := `Nat.le_pred_l` (*compat* "8.4"). **Notation** `le_pred` := `Nat.pred_le_mono` (*compat* "8.4").

Hint `Resolve` `le_pred_n` : *arith v62*.

56.5 A different elimination principle for the order on natural numbers

Lemma `le_elim_rel` :

$\forall P:\mathbf{nat} \rightarrow \mathbf{nat} \rightarrow \mathbf{Prop},$
 $(\forall p, P\ 0\ p) \rightarrow$
 $(\forall p\ (q:\mathbf{nat}), p \leq q \rightarrow P\ p\ q \rightarrow P\ (\mathbf{S}\ p)\ (\mathbf{S}\ q)) \rightarrow$
 $\forall n\ m, n \leq m \rightarrow P\ n\ m.$

Chapter 57

Library **Coq.Arith.Lt**

Strict order on natural numbers.

This file is mostly OBSOLETE now, see module *PeanoNat.Nat* instead.

lt is defined in library *Init/Peano.v* as:

Definition `lt` (`n m:nat`) := `S n <= m`.

Infix "`<`" := `lt` : `nat_scope`.

Require Import `PeanoNat`.

Local Open Scope `nat_scope`.

57.1 Irreflexivity

Notation `lt_irrefl` := `Nat.lt_irrefl (compat "8.4")`.

Hint Resolve `lt_irrefl`: *arith v62*.

57.2 Relationship between *le* and *lt*

Theorem `lt_le_S` `n m` : `n < m` → `S n ≤ m`.

Theorem `lt_n_Sm_le` `n m` : `n < S m` → `n ≤ m`.

Theorem `le_lt_n_Sm` `n m` : `n ≤ m` → `n < S m`.

Hint Immediate `lt_le_S`: *arith v62*.

Hint Immediate `lt_n_Sm_le`: *arith v62*.

Hint Immediate `le_lt_n_Sm`: *arith v62*.

Theorem `le_not_lt` `n m` : `n ≤ m` → ¬ `m < n`.

Theorem `lt_not_le` `n m` : `n < m` → ¬ `m ≤ n`.

Hint Immediate `le_not_lt lt_not_le`: *arith v62*.

57.3 Asymmetry

Notation `lt_asym` := `Nat.lt_asymm` (*compat* "8.4").

57.4 Order and 0

Notation `lt_0_Sn` := `Nat.lt_0_succ` (*compat* "8.4"). **Notation** `lt_n_0` := `Nat.nlt_0_r` (*compat* "8.4").

Theorem `neq_0_lt` $n : 0 \neq n \rightarrow 0 < n$.

Theorem `lt_0_neq` $n : 0 < n \rightarrow 0 \neq n$.

Hint `Resolve` `lt_0_Sn` `lt_n_0` : *arith v62*.

Hint `Immediate` `neq_0_lt` `lt_0_neq` : *arith v62*.

57.5 Order and successor

Notation `lt_n_Sn` := `Nat.lt_succ_diag_r` (*compat* "8.4"). **Notation** `lt_S` := `Nat.lt_lt_succ_r` (*compat* "8.4").

Theorem `lt_n_S` $n\ m : n < m \rightarrow S\ n < S\ m$.

Theorem `lt_S_n` $n\ m : S\ n < S\ m \rightarrow n < m$.

Hint `Resolve` `lt_n_Sn` `lt_S` `lt_n_S` : *arith v62*.

Hint `Immediate` `lt_S_n` : *arith v62*.

57.6 Predecessor

Lemma `S_pred` $n\ m : m < n \rightarrow n = S\ (\text{pred}\ n)$.

Lemma `lt_pred` $n\ m : S\ n < m \rightarrow n < \text{pred}\ m$.

Lemma `lt_pred_n_n` $n : 0 < n \rightarrow \text{pred}\ n < n$.

Hint `Immediate` `lt_pred` : *arith v62*.

Hint `Resolve` `lt_pred_n_n` : *arith v62*.

57.7 Transitivity properties

Notation `lt_trans` := `Nat.lt_trans` (*compat* "8.4").

Notation `lt_le_trans` := `Nat.lt_le_trans` (*compat* "8.4").

Notation `le_lt_trans` := `Nat.le_lt_trans` (*compat* "8.4").

Hint `Resolve` `lt_trans` `lt_le_trans` `le_lt_trans` : *arith v62*.

57.8 Large = strict or equal

Notation `le_lt_or_eq_iff` := `Nat.lt_eq_cases` (*compat* "8.4").

Theorem `le_lt_or_eq` $n\ m : n \leq m \rightarrow n < m \vee n = m$.

Notation `lt_le_weak` := `Nat.lt_le_incl` (*compat* "8.4").

Hint Immediate `lt_le_weak`: *arith v62*.

57.9 Dichotomy

Notation `le_or_lt` := `Nat.le_gt_cases` (*compat* "8.4").

Theorem `nat_total_order` $n\ m : n \neq m \rightarrow n < m \vee m < n$.

For compatibility, we “Require” the same files as before

Require Import `Le`.

Chapter 58

Library **Coq.Arith.Max**

THIS FILE IS DEPRECATED. Use *PeanoNat.Nat* instead.

Require Import PeanoNat.

Local Open Scope *nat_scope*.

Implicit Types *m n p* : **nat**.

Notation *max* := **Nat.max** (*only parsing*).

Definition *max_0_l* := **Nat.max_0_l**.

Definition *max_0_r* := **Nat.max_0_r**.

Definition *succ_max_distr* := **Nat.succ_max_distr**.

Definition *plus_max_distr_l* := **Nat.add_max_distr_l**.

Definition *plus_max_distr_r* := **Nat.add_max_distr_r**.

Definition *max_case_strong* := **Nat.max_case_strong**.

Definition *max_spec* := **Nat.max_spec**.

Definition *max_dec* := **Nat.max_dec**.

Definition *max_case* := **Nat.max_case**.

Definition *max_idempotent* := **Nat.max_id**.

Definition *max_assoc* := **Nat.max_assoc**.

Definition *max_comm* := **Nat.max_comm**.

Definition *max_l* := **Nat.max_l**.

Definition *max_r* := **Nat.max_r**.

Definition *le_max_l* := **Nat.le_max_l**.

Definition *le_max_r* := **Nat.le_max_r**.

Definition *max_lub_l* := **Nat.max_lub_l**.

Definition *max_lub_r* := **Nat.max_lub_r**.

Definition *max_lub* := **Nat.max_lub**.

Hint Resolve

Nat.max_l Nat.max_r Nat.le_max_l Nat.le_max_r : arith v62.

Hint Resolve

Nat.min_l Nat.min_r Nat.le_min_l Nat.le_min_r : arith v62.

Chapter 59

Library **Coq.Arith.Minus**

Properties of subtraction between natural numbers.

This file is mostly OBSOLETE now, see module *PeanoNat.Nat* instead.

minus is now an alias for *Nat.sub*, which is defined in *Init/Nat.v* as:

```
Fixpoint sub (n m:nat) : nat :=
  match n, m with
  | S k, S l => k - l
  | _, _ => n
  end
where "n - m" := (sub n m) : nat_scope.
```

Require Import PeanoNat Lt Le.

Local Open Scope *nat_scope*.

59.1 0 is right neutral

Lemma *minus_n_0* $n : n = n - 0$.

59.2 Permutation with successor

Lemma *minus_Sn_m* $n m : m \leq n \rightarrow S (n - m) = S n - m$.

Theorem *pred_of_minus* $n : \text{pred } n = n - 1$.

59.3 Diagonal

Notation *minus_diag* := *Nat.sub_diag* (*compat* "8.4").

Lemma *minus_diag_reverse* $n : 0 = n - n$.

Notation *minus_n_n* := *minus_diag_reverse*.

59.4 Simplification

Lemma `minus_plus_simpl_l_reverse` $n\ m\ p : n - m = p + n - (p + m)$.

59.5 Relation with plus

Lemma `plus_minus` $n\ m\ p : n = m + p \rightarrow p = n - m$.

Lemma `minus_plus` $n\ m : n + m - n = m$.

Lemma `le_plus_minus_r` $n\ m : n \leq m \rightarrow n + (m - n) = m$.

Lemma `le_plus_minus` $n\ m : n \leq m \rightarrow m = n + (m - n)$.

59.6 Relation with order

Notation `minus_le_compat_r` :=
 `Nat.sub_le_mono_r` (*compat* "8.4").

Notation `minus_le_compat_l` :=
 `Nat.sub_le_mono_l` (*compat* "8.4").

Notation `le_minus` := `Nat.le_sub_l` (*compat* "8.4"). **Notation** `lt_minus` := `Nat.sub_lt` (*compat* "8.4").

Lemma `lt_O_minus_lt` $n\ m : 0 < n - m \rightarrow m < n$.

Theorem `not_le_minus_0` $n\ m : \neg m \leq n \rightarrow n - m = 0$.

59.7 Hints

Hint `Resolve` *minus_n_O*: *arith v62*.

Hint `Resolve` *minus_Sn_m*: *arith v62*.

Hint `Resolve` *minus_diag_reverse*: *arith v62*.

Hint `Resolve` *minus_plus_simpl_l_reverse*: *arith v62*.

Hint `Immediate` *plus_minus*: *arith v62*.

Hint `Resolve` *minus_plus*: *arith v62*.

Hint `Resolve` *le_plus_minus*: *arith v62*.

Hint `Resolve` *le_plus_minus_r*: *arith v62*.

Hint `Resolve` *lt_minus*: *arith v62*.

Hint `Immediate` *lt_O_minus_lt*: *arith v62*.

Chapter 60

Library **Coq.Arith.Min**

THIS FILE IS DEPRECATED. Use *PeanoNat.Nat* instead.

```
Require Import PeanoNat.

Local Open Scope nat_scope.
Implicit Types m n p : nat.

Notation min := Nat.min (only parsing).

Definition min_0_l := Nat.min_0_l.
Definition min_0_r := Nat.min_0_r.
Definition succ_min_distr := Nat.succ_min_distr.
Definition plus_min_distr_l := Nat.add_min_distr_l.
Definition plus_min_distr_r := Nat.add_min_distr_r.
Definition min_case_strong := Nat.min_case_strong.
Definition min_spec := Nat.min_spec.
Definition min_dec := Nat.min_dec.
Definition min_case := Nat.min_case.
Definition min_idempotent := Nat.min_id.
Definition min_assoc := Nat.min_assoc.
Definition min_comm := Nat.min_comm.
Definition min_l := Nat.min_l.
Definition min_r := Nat.min_r.
Definition le_min_l := Nat.le_min_l.
Definition le_min_r := Nat.le_min_r.
Definition min_glb_l := Nat.min_glb_l.
Definition min_glb_r := Nat.min_glb_r.
Definition min_glb := Nat.min_glb.
```


Chapter 61

Library **Coq.Arith.Mult**

61.1 Properties of multiplication.

This file is mostly OBSOLETE now, see module *PeanoNat.Nat* instead.

Nat.mul is defined in *Init/Nat.v*.

Require Import PeanoNat.

For Compatibility: **Require Export** Plus Minus Le Lt.

Local Open Scope *nat_scope*.

61.2 *nat* is a semi-ring

61.2.1 Zero property

Notation *mult_0_l* := *Nat.mul_0_l* (*compat* "8.4"). **Notation** *mult_0_r* := *Nat.mul_0_r* (*compat* "8.4").

61.2.2 1 is neutral

Notation *mult_1_l* := *Nat.mul_1_l* (*compat* "8.4"). **Notation** *mult_1_r* := *Nat.mul_1_r* (*compat* "8.4").

Hint Resolve *mult_1_l mult_1_r*: *arith v62*.

61.2.3 Commutativity

Notation *mult_comm* := *Nat.mul_comm* (*compat* "8.4").

Hint Resolve *mult_comm*: *arith v62*.

61.2.4 Distributivity

Notation *mult_plus_distr_r* :=
Nat.mul_add_distr_r (*compat* "8.4").

```

Notation mult_plus_distr_l :=
  Nat.mul_add_distr_l (compat "8.4").
Notation mult_minus_distr_r :=
  Nat.mul_sub_distr_r (compat "8.4").
Notation mult_minus_distr_l :=
  Nat.mul_sub_distr_l (compat "8.4").
Hint Resolve mult_plus_distr_r: arith v62.
Hint Resolve mult_minus_distr_r: arith v62.
Hint Resolve mult_minus_distr_l: arith v62.

```

61.2.5 Associativity

```

Notation mult_assoc := Nat.mul_assoc (compat "8.4").
Lemma mult_assoc_reverse n m p : n × m × p = n × (m × p).
Hint Resolve mult_assoc_reverse: arith v62.
Hint Resolve mult_assoc: arith v62.

```

61.2.6 Inversion lemmas

```

Lemma mult_is_O n m : n × m = 0 → n = 0 ∨ m = 0.
Lemma mult_is_one n m : n × m = 1 → n = 1 ∧ m = 1.

```

61.2.7 Multiplication and successor

```

Notation mult_succ_l := Nat.mul_succ_l (compat "8.4"). Notation mult_succ_r := Nat.mul_succ_r
(compat "8.4").

```

61.3 Compatibility with orders

```

Lemma mult_O_le n m : m = 0 ∨ n ≤ m × n.
Hint Resolve mult_O_le: arith v62.
Lemma mult_le_compat_l n m p : n ≤ m → p × n ≤ p × m.
Hint Resolve mult_le_compat_l: arith.
Lemma mult_le_compat_r n m p : n ≤ m → n × p ≤ m × p.
Lemma mult_le_compat n m p q : n ≤ m → p ≤ q → n × p ≤ m × q.
Lemma mult_S_lt_compat_l n m p : m < p → S n × m < S n × p.
Hint Resolve mult_S_lt_compat_l: arith.
Lemma mult_lt_compat_l n m p : n < m → 0 < p → p × n < p × m.
Lemma mult_lt_compat_r n m p : n < m → 0 < p → n × p < m × p.
Lemma mult_S_le_reg_l n m p : S n × m ≤ S n × p → m ≤ p.

```

61.4 $n|->2*n$ and $n|->2n+1$ have disjoint image

Theorem `odd_even_lem` $p\ q : 2 \times p + 1 \neq 2 \times q$.

61.5 Tail-recursive mult

tail_mult is an alternative definition for *mult* which is tail-recursive, whereas *mult* is not. This can be useful when extracting programs.

```
Fixpoint mult_acc (s:nat) m n : nat :=  
  match n with  
  | 0 => s  
  | S p => mult_acc (tail_plus m s) m p  
  end.
```

Lemma `mult_acc_aux` : $\forall\ n\ m\ p, m + n \times p = \text{mult_acc}\ m\ p\ n$.

Definition `tail_mult` $n\ m := \text{mult_acc}\ 0\ m\ n$.

Lemma `mult_tail_mult` : $\forall\ n\ m, n \times m = \text{tail_mult}\ n\ m$.

TailSimpl transforms any *tail_plus* and *tail_mult* into *plus* and *mult* and simplify

```
Ltac tail_simpl :=  
  repeat rewrite <- plus_tail_plus; repeat rewrite <- mult_tail_mult;  
  simpl.
```

Chapter 62

Library `Coq.Arith.Peano_dec`

```
Require Import Decidable PeanoNat.
Require Eqdep_dec.
Local Open Scope nat_scope.
Implicit Types m n x y : nat.
Theorem O_or_S n : {m : nat | S m = n} + {0 = n}.
Notation eq_nat_dec := Nat.eq_dec (compat "8.4").
Hint Resolve O_or_S eq_nat_dec: arith.
Theorem dec_eq_nat n m : decidable (n = m).
Definition UIP_nat:= Eqdep_dec.UIP_dec Nat.eq_dec.
Import EqNotations.
Lemma le_unique:  $\forall m n (le\_mn1\ le\_mn2 : m \leq n), le\_mn1 = le\_mn2$ .
  For compatibility Require Import Le Lt.
```

Chapter 63

Library **Coq.Arith.Plus**

Properties of addition.

This file is mostly OBSOLETE now, see module *PeanoNat.Nat* instead.

Nat.add is defined in *Init/Nat.v* as:

```
Fixpoint add (n m:nat) : nat :=
  match n with
  | 0 => m
  | S p => S (p + m)
  end
where "n + m" := (add n m) : nat_scope.
```

Require Import PeanoNat.

Local Open Scope *nat_scope*.

63.1 Neutrality of 0, commutativity, associativity

Notation *plus_0_l* := *Nat.add_0_l* (*compat* "8.4").

Notation *plus_0_r* := *Nat.add_0_r* (*compat* "8.4").

Notation *plus_comm* := *Nat.add_comm* (*compat* "8.4").

Notation *plus_assoc* := *Nat.add_assoc* (*compat* "8.4").

Notation *plus_permute* := *Nat.add_shuffle3* (*compat* "8.4").

Definition *plus_Snm_nSm* : $\forall n\ m, S\ n + m = n + S\ m :=$
Peano.plus_n_Sm.

Lemma *plus_assoc_reverse* *n m p* : $n + m + p = n + (m + p)$.

63.2 Simplification

Lemma *plus_reg_l* *n m p* : $p + n = p + m \rightarrow n = m$.

Lemma *plus_le_reg_l* *n m p* : $p + n \leq p + m \rightarrow n \leq m$.

Lemma *plus_lt_reg_l* *n m p* : $p + n < p + m \rightarrow n < m$.

63.3 Compatibility with order

Lemma `plus_le_compat_l` $n\ m\ p : n \leq m \rightarrow p + n \leq p + m$.
 Lemma `plus_le_compat_r` $n\ m\ p : n \leq m \rightarrow n + p \leq m + p$.
 Lemma `plus_lt_compat_l` $n\ m\ p : n < m \rightarrow p + n < p + m$.
 Lemma `plus_lt_compat_r` $n\ m\ p : n < m \rightarrow n + p < m + p$.
 Lemma `plus_le_compat` $n\ m\ p\ q : n \leq m \rightarrow p \leq q \rightarrow n + p \leq m + q$.
 Lemma `plus_le_lt_compat` $n\ m\ p\ q : n \leq m \rightarrow p < q \rightarrow n + p < m + q$.
 Lemma `plus_lt_le_compat` $n\ m\ p\ q : n < m \rightarrow p \leq q \rightarrow n + p < m + q$.
 Lemma `plus_lt_compat` $n\ m\ p\ q : n < m \rightarrow p < q \rightarrow n + p < m + q$.
 Lemma `le_plus_l` $n\ m : n \leq n + m$.
 Lemma `le_plus_r` $n\ m : m \leq n + m$.
 Theorem `le_plus_trans` $n\ m\ p : n \leq m \rightarrow n \leq m + p$.
 Theorem `lt_plus_trans` $n\ m\ p : n < m \rightarrow n < m + p$.

63.4 Inversion lemmas

Lemma `plus_is_O` $n\ m : n + m = 0 \rightarrow n = 0 \wedge m = 0$.
 Definition `plus_is_one` $m\ n :$
 $m + n = 1 \rightarrow \{m = 0 \wedge n = 1\} + \{m = 1 \wedge n = 0\}$.

63.5 Derived properties

Notation `plus_permute_2_in_4` $:= \text{Nat.add_shuffle1 } (\text{compat } "8.4")$.

63.6 Tail-recursive plus

`tail_plus` is an alternative definition for `plus` which is tail-recursive, whereas `plus` is not. This can be useful when extracting programs.

Fixpoint `tail_plus` $n\ m : \text{nat} :=$
`match` n `with`
`| O` $\Rightarrow m$
`| S n` $\Rightarrow \text{tail_plus } n\ (\text{S } m)$
`end`.
 Lemma `plus_tail_plus` $: \forall\ n\ m, n + m = \text{tail_plus } n\ m$.

63.7 Discrimination

Lemma `succ_plus_discr` $n\ m : n \neq S\ (m+n)$.

Lemma `n_SS` $n : n \neq S\ (S\ n)$.

Lemma `n_SSS` $n : n \neq S\ (S\ (S\ n))$.

Lemma `n_SSSS` $n : n \neq S\ (S\ (S\ (S\ n)))$.

63.8 Compatibility Hints

Hint Immediate `plus_comm` : *arith v62*.

Hint Resolve `plus_assoc` `plus_assoc_reverse` : *arith v62*.

Hint Resolve `plus_le_compat_l` `plus_le_compat_r` : *arith v62*.

Hint Resolve `le_plus_l` `le_plus_r` `le_plus_trans` : *arith v62*.

Hint Immediate `lt_plus_trans` : *arith v62*.

Hint Resolve `plus_lt_compat_l` `plus_lt_compat_r` : *arith v62*.

For compatibility, we “Require” the same files as before

Require Import `Le Lt`.

Chapter 64

Library `Coq.Arith.Wf_nat`

Well-founded relations and natural numbers

```
Require Import PeanoNat Lt.
```

```
Local Open Scope nat_scope.
```

```
Implicit Types m n p : nat.
```

```
Section Well_founded_Nat.
```

```
Variable A : Type.
```

```
Variable f : A → nat.
```

```
Definition ltof (a b:A) := f a < f b.
```

```
Definition gtof (a b:A) := f b > f a.
```

```
Theorem well_founded_ltof : well_founded ltof.
```

```
Theorem well_founded_gtof : well_founded gtof.
```

It is possible to directly prove the induction principle going back to primitive recursion on natural numbers (*induction_ltof1*) or to use the previous lemmas to extract a program with a fixpoint (*induction_ltof2*)

the ML-like program for *induction_ltof1* is :

```
let induction_ltof1 f F a =
```

```
  let rec indrec n k =
```

```
    match n with
```

```
    | O → error
```

```
    | S m → F k (indrec m)
```

```
  in indrec (f a + 1) a
```

the ML-like program for *induction_ltof2* is :

```
let induction_ltof2 F a = indrec a
```

```
where rec indrec a = F a indrec;;
```

```
Theorem induction_ltof1 :
```

```
  ∀ P:A → Set,
```

```
  (∀ x:A, (∀ y:A, ltof y x → P y) → P x) → ∀ a:A, P a.
```

```
Theorem induction_gtof1 :
```


$\forall P:A \rightarrow \text{Set},$
 $(\forall x:A, (\forall y:A, \text{gtof } y \ x \rightarrow P \ y) \rightarrow P \ x) \rightarrow \forall a:A, P \ a.$

Theorem `induction_ltof2` :

$\forall P:A \rightarrow \text{Set},$
 $(\forall x:A, (\forall y:A, \text{ltof } y \ x \rightarrow P \ y) \rightarrow P \ x) \rightarrow \forall a:A, P \ a.$

Theorem `induction_gtof2` :

$\forall P:A \rightarrow \text{Set},$
 $(\forall x:A, (\forall y:A, \text{gtof } y \ x \rightarrow P \ y) \rightarrow P \ x) \rightarrow \forall a:A, P \ a.$

If a relation R is compatible with lt i.e. if $x \ R \ y \Rightarrow f(x) < f(y)$ then R is well-founded.

Variable $R : A \rightarrow A \rightarrow \text{Prop}.$

Hypothesis $H_compat : \forall x \ y:A, R \ x \ y \rightarrow f \ x < f \ y.$

Theorem `well_founded_lt_compat` : `well_founded` $R.$

End `Well_founded_Nat.`

Lemma `lt_wf` : `well_founded` $lt.$

Lemma `lt_wf_rec1` :

$\forall n \ (P:\text{nat} \rightarrow \text{Set}), (\forall n, (\forall m, m < n \rightarrow P \ m) \rightarrow P \ n) \rightarrow P \ n.$

Lemma `lt_wf_rec` :

$\forall n \ (P:\text{nat} \rightarrow \text{Set}), (\forall n, (\forall m, m < n \rightarrow P \ m) \rightarrow P \ n) \rightarrow P \ n.$

Lemma `lt_wf_ind` :

$\forall n \ (P:\text{nat} \rightarrow \text{Prop}), (\forall n, (\forall m, m < n \rightarrow P \ m) \rightarrow P \ n) \rightarrow P \ n.$

Lemma `gt_wf_rec` :

$\forall n \ (P:\text{nat} \rightarrow \text{Set}), (\forall n, (\forall m, n > m \rightarrow P \ m) \rightarrow P \ n) \rightarrow P \ n.$

Lemma `gt_wf_ind` :

$\forall n \ (P:\text{nat} \rightarrow \text{Prop}), (\forall n, (\forall m, n > m \rightarrow P \ m) \rightarrow P \ n) \rightarrow P \ n.$

Lemma `lt_wf_double_rec` :

$\forall P:\text{nat} \rightarrow \text{nat} \rightarrow \text{Set},$
 $(\forall n \ m,$
 $(\forall p \ q, p < n \rightarrow P \ p \ q) \rightarrow$
 $(\forall p, p < m \rightarrow P \ n \ p) \rightarrow P \ n \ m) \rightarrow \forall n \ m, P \ n \ m.$

Lemma `lt_wf_double_ind` :

$\forall P:\text{nat} \rightarrow \text{nat} \rightarrow \text{Prop},$
 $(\forall n \ m,$
 $(\forall p \ (q:\text{nat}), p < n \rightarrow P \ p \ q) \rightarrow$
 $(\forall p, p < m \rightarrow P \ n \ p) \rightarrow P \ n \ m) \rightarrow \forall n \ m, P \ n \ m.$

Hint `Resolve` lt_wf : *arith*.

Hint `Resolve` $well_founded_lt_compat$: *arith*.

Section `LT_WF_REL.`

Variable $A : \text{Set}.$

Variable $R : A \rightarrow A \rightarrow \text{Prop}.$

Variable $F : A \rightarrow \text{nat} \rightarrow \text{Prop}.$

```

Definition inv_lt_rel x y := exists2 n, F x n & (∀ m, F y m → n < m).
Hypothesis F_compat : ∀ x y:A, R x y → inv_lt_rel x y.
Remark acc_lt_rel : ∀ x:A, (∃ n, F x n) → Acc R x.

Theorem well_founded_inv_lt_rel_compat : well_founded R.
End LT_WF_REL.

Lemma well_founded_inv_rel_inv_lt_rel :
  ∀ (A:Set) (F:A → nat → Prop), well_founded (inv_lt_rel A F).

  A constructive proof that any non empty decidable subset of natural numbers has a least element

Set Implicit Arguments.
Require Import Le.
Require Import Compare_dec.
Require Import Decidable.

Definition has_unique_least_element (A:Type) (R:A→A→Prop) (P:A→Prop) :=
  ∃! x, P x ∧ ∀ x', P x' → R x x'.

Lemma dec_inh_nat_subset_has_unique_least_element :
  ∀ P:nat→Prop, (∀ n, P n ∨ ¬ P n) →
    (∃ n, P n) → has_unique_least_element le P.

Unset Implicit Arguments.

Notation iter_nat n A f x := (nat_rect (fun _ ⇒ A) x (fun _ ⇒ f) n) (only parsing).

```